member from each of the colleges represented, was appointed to study the teaching of mathematics in the high schools and colleges of the southeast, and to make a comprehensive report of their findings.

The next annual meeting will be held at the University of Georgia.

On the evening of the 18th, a dinner in Primrose Grill was held in honor of the visiting speaker, Professor Dunham Jackson. At this time vice-Chairman T. M. Simpson presided, and President J. J. Tigert of the University of Florida gave the address of welcome, while Dean Floyd Field of Georgia School of Technology responded.

Papers were presented as follows:

1. "How can a high school teacher advance professionally?" by Professor F. W. Kokomoor, University of Florida.
2. "The value of college mathematics to the high school teacher" by Professor Dunham Jackson, University of Minnesota.
3. "Problems in the teaching of high school mathematics" by D. M. Bernard, Robt. E. Lee High School, Jacksonville, Fla., by invitation.
4. "Problems in the teaching of freshman college mathematics" by Professor T. M. Simpson, University of Florida.
5. "Some observations on the correlation of these two problems" by U. P. Davis, University of Florida.
6. "The study of mathematics" by Professor Jackson.
7. "A conference method" by Professor E. F. Weinberg, Rollins College.
8. "Solution of a certain transcendental equation" by Professor D. F. Barrow, University of Georgia.
9. "Frequency functions" by Professor Jackson.
10. "Graphical determination of moments of inertia" by H. H. Germond, University of Florida, by invitation.
11. "Remarks on non-Aristotelian mathematics" by B. F. Dostal, University of Florida.
12. "Calculations with approximate data" by Professor C. G. Phipps, University of Florida, by invitation.

Abstracts of some of these papers follow:

1. Professor Kokomoor emphasized five points, as follows: (1) Strenouus application to the work until one is born into the spirit of teaching when one's interest becomes contagious and students are infected. (2) The development of an outlook from which pupils can be shown how almost every field of human endeavor is shot through and through with mathematics. (3) The study of order, so that there may be adequacy in the selection of material, in the manner of presenting it, in one's own method of study and of directing students in their advance. (4) The acquisition of more of the content of mathematics so as to keep apace in the march of knowledge. (5) The spirit of cooperation and the exercise of initiative in seeing and satisfying the needs of the department.

3. In discussing Algebra Mr. Bernard's remarks centered around the formation and solution of the equation, and the symbols were presented as forming a powerful system of shorthand. The use of charts in geometry was discussed at length and a method of teaching pupils to prove theorems was presented. Indirect proofs, loci, the Pythagorean theorem and similar figures were discussed as important subjects which present difficulty and suggestions were offered for treating these topics. In solid geometry it was recommended that theorems of minor importance and those with long and difficult proofs be treated as assumptions if necessary to give ample time for a thorough study of numerical exercises.

4. Professor Simpson discussed the following questions: How can we stimulate scholarship among freshman? What shall we teach in a beginning course? Is there any best method of teaching mathematics? What are some of the deficiencies in the preparation of freshmen?

5. One of the principal causes of the unsatisfactory preparation in mathematics for college entrance is the lack of a definite objective on the part of the high school teacher. Modern conditions demand that college preparation should not be this objective. This need should be met by enriching the course of study for the better students only, in such a way that, for prospective college students, the foundations in mathematics will be strengthened. Every accredited high school should issue two different grades of diplomas, one of which does not admit the holder to college.

8. In this paper, Professor Barrow sets up an infinite exponential which satisfies the equation $x \log y = y \log x$ and whose graph consists of a segment of a straight line joined at right angles by a curve which approaches an asymptote parallel to the x-axis and one unit above it.

10. If an area enclosed by a curve in the x, y plane be remapped by the transformation $u = x$, $v = y^2/2$, the area enclosed by the new curve (in the u, v plane) will numerically equal the moment of the original area about the x-axis. The transformation $u = x$, $v = y^3/3$, results in an area in the u, v plane equal to the moment of inertia about the x-axis of the original area in the x, y plane. The area obtained by remapping in a u, v plane an area in the x, y plane by the transformation $u = x$, $v = \pi y^4/2$, is equal to the moment of inertia about the axis of rotation of the volume formed by rotating the original area about the x-axis. Remapping a curve from the r, θ plane to the plane $\rho = r^2/\sqrt{2}$, $\phi = \theta$, the area enclosed by the new curve equals the polar moment of inertia of the area enclosed by the original curve.

The use of suitably prepared coordinate papers facilitates the application of this method of determining the moment of inertia.

11. Professor Dostal discussed the bearing of the work of the Polish School of Intuitionistic Formalists upon the questions raised by Eddington in his address of January 4, 1932, (reported in London Nature, Feb. 13). According to Eddington, we have already reached the stage, in Theoretical Physics, where it is considered desirable to abandon Determinism, at least for the time being.

However, even Eddington admits the desirability, for certain practical purposes, of using Deterministic means as a sort of "symbolic operational" method for obtaining results quickly and conveniently. For astronomical purposes, for example, we could continue to use what may be called, perhaps, "Astronomical Determinism," to calculate results, whereas for Electron Physics we could make use of the "Indeterministic" or "Secondary " Laws (as Eddington terms them), that is, the Probability Methods of the New Quantum Mechanics. Between these two extreme cases, however, there would be a set of intermediate cases And this would lead us into logical difficulties, for the notion of Probability has no recognized place in classical Deductive Logic. Furthermore, Determinism enters the scientific arena mainly through the door of Deductive Logic, and if we are to modify Determinism we will have to modify our whole system of thinking. The necessity for such a fundamental revision of our methods of reasoning is strongly suggested in Eddington's statement that while naive realism, materialism, and the Mechanistic Hypothesis were simple, yet it was only by closing our eyes to the essential nature of experience that they could be made to seem credible. Suppose now, it were possible to construct a multiple-valued Logical System (instead of the simple Two-valued System of classical Logic, as exemplified, for instance, in Whitehead and Russell's Principia Mathematica) of such a sort that it could constitute a valid and satisfactory basis for a whole set of distinct Theories of Sets; then the way would be open for the introduction of the notion of Probability into Deductive Logic, and the logical difficulties mentioned above might be smoothed out. Now, as a result of the work of Chwistek (Mathematische Zeitschrift, Vols. 26 and 30), and of Lesniewski, Tarski, Lukasiewicz, Greniewski, and others, together with the fundamental work of Korzybski (a forthcoming book by him will explain the work of the Polish School), a start in this direction has already been made, and a multi-valued system has actually been constructed, and looks to be very promising for the purposes mentioned.

12. This paper discusses the process of "rounding off" as applied to numbers, the absolute and relative errors in sums, differences, products and quotients. Mention is made of the accuracy in connection with logarithms and trigonometric functions. The reverse problem is taken up, namely, the allowable error in the data if there is a certain desired accuracy in the result.

W. W. RANKIN, *Secretary*

THE NINTH ANNUAL MEETING OF THE INDIANA SECTION

The ninth annual meeting of the Indiana Section of the Mathematical Association of America was held on Friday and Saturday, May 6, and 7, 1932, at Butler University, Indianapolis, Indiana.

There were sixty present at the meeting including the following twenty-six members of the Association:

Gladys L. Banes, G. A. Bliss, Stanley Bolks, G. E. Carscallen, H. T. Davis, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, F. H. Hodge, H. K. Hughes, E. N. Johnson, E. L. Klinger, Juna M. Lutz, T. E. Mason, H. A. Meyer, T. W. Moore, Eunice C. Orr, Saul Pollock, J. A. Reising, C. K. Robbins, L. S. Shively, R. O. Virts, K. P. Williams.

On Friday afternoon at 5:30 a reception was given to the visiting members and their guests which was followed by a banquet attended by fifty-five persons. Dean J. W. Putnam of Butler University officiated as toast master and made an address of welcome to the members of the Section and their guests. Responses were made by Professor Carscallen of Wabash College, chairman of the Section, Professor Edington of De Pauw University and Professor Cora B. Hennel of Indiana University.

At 8:15 a public lecture was given by Professor G. A. Bliss of the University of Chicago on the subject, "The Structures of Pure and Applied Mathematical Sciences." In this address Professor Bliss characterized pure mathematical science as consisting of postulates, definitions, and theorems. The theory of the real number system based upon four simple postulates for positive integers was cited as an example. The structure of an applied mathematical science is similar except that the usefulness of the theory depends upon the accuracy with which the postulates correspond to simple observed data, and the logical conclusions of the theory to the results of more complicated observations. It is not true that there is a unique mathematical theory for the correlation of a particular set of natural phenomena. On account of the looseness of the fit between theory and observation, which is always present, a multiplicity of theories for the correlation of the same set of observed data is always possible. Euclidean and non-Euclidean theories of plane geometry, the Ptolemaic and Copernican theories in astronomy, and the various quantum theories were described briefly as illustrations.

Following the address of Professor Bliss, Professor Saul Pollock, of the Indiana State Teachers' College at Terre Haute, gave a public exhibition of "Skew Curve Projection." This exhibition consisted in the creation of space curves by throwing light upon string models of various types of surfaces. By means of the device of photographing curves of high order and using these lantern slides in turn to generate new space curves, it was possible to obtain curves of remarkably high orders.

At the session on Saturday morning, presided over by Professor G. E. Carscallen, Wabash College, chairman, the following officers were elected: Professor K. P. Williams, Indiana University, Chairman; Professor Juna M. Lutz, Butler University, vice-Chairman; Professor H. T. Davis, Indiana University, Secretary-Treasurer.

A chairman's address was made by Professor G. E. Carscallen on the subject, "The Pathology of Mathematics." In this address Professor Carscallen called attention to the fact that "educational research" during the past two or

three decades has encroached by leaps and bounds upon other departments in college and university curricula. This encroachment presents to mathematics in particular an especial menace, since too many people, ignorant of its aims and unappreciative of its importance, have attempted foolish modifications of the mathematics courses both in intermediate and college teaching. Present low standards of attainment by graduates are attributable to these causes. The speaker cited the lowered standards of the North Central Association with regard to mathematics as an evidence of this dangerous trend. Professor Carscallen particularly urged the members of the Indiana Section to take a more vigorous part in the framing of curricula and in other activities where the cause of mathematics could be more effectively defended.

By special invitation Dr. Cornelius Lanczos of Frankfort University, Germany, and Purdue University, made an hour's address on the subject, "An Elementary Development of Riemannian Geometry with Application to Relativity." In this address Dr. Lanczos considered the geometry of the line element, $ds^2 = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2$, and showed how interpretations in Euclidean geometry could be generalized in the Riemannian and Lobachevskian cases. Making use of the variation principle applied to the line integral $I = \int ds$, the speaker showed how the mechanics of the Keplerian orbits could be obtained as an interpretation of the geometrical picture. Proceeding from this elegant discussion, the speaker carried the generalization into four dimensions and showed how the mechanics of the Einstein physics came as a natural consequence of the geometrical considerations.

The remainder of the program consisted of the following papers. Due to illness Professor Heath was unable to give his paper, but was represented by George Manning of Franklin College.

1. "A study in Keplerian elliptic motion" by M. Wiles Keller, Indiana University, by invitation.

2. "Functions analogous to Hermite polynomials in the problem of curve fitting" by Professor H. E. H. Greenleaf, De Pauw University.

3. "Language, logic, and mathematics" by Dr. A. F. Bentley, Paoli, Indiana, by invitation.

4. "Maxima and minima of radii of curves" by Professor F. H. Hodge, Purdue University.

5. "Some composite polyhedrons" by Professor D. H. Heath, Franklin College.

6. "A new technique in the analysis of trend lines" by Professor H. T. Davis, Indiana University.

7. "Higher geometry in the college curriculum" by Professor J. E. Dotterer, Manchester College.

8. "Some elementary geometrical applications of group theory" by Professor W. E. Edington, De Pauw University.

Abstracts of the papers follow, the numbers corresponding to the list of titles.

1. In this paper the expression for the angular velocity about a point between the foci in Keplerian elliptic motion was determined and the cubic equation derived, the roots of which give the values for which the angular velocity is a maximum and a minimum. Some of the properties of this cubic, the coefficients of which depend upon two parameters, were given and their relationship to the angular velocity noted.

2. Professor Greenleaf discussed the least-square approximation to data in which the variates are equally spaced and the observed frequencies are given the binomial coefficients as weighting factors. By a method parallel to the Gram-Charlier development of continuous variates in the type-A curve, a set of functions analogous to the Hermite polynomials were found for discrete variates. These functions have a generating function and a recurrence formula; each satisfies a given difference equation of second order, and has an orthogonality property similar to that of the Hermite polynomials. The coefficients to be used with these functions are computed and tabulated, permitting the determination of the least-square equation with a minimum amount of computation.

At the close of the paper Professor E. H. Hildebrandt of De Pauw University discussed these functions showing how they fit into the general system of polynomials connected with the Charlier expansion, using his paper published in the Annals of Mathematical Statistics, November, 1931, as the basis of his discussion.

3. Dr. Bentley undertook to show the manner in which language, logic and mathematics may be inspected as differentiated aspects of that historical field of human behavior, indicated by the word "knowledge" and by various associated terms. For the investigation of this common field a procedure to be called "Semantic Analysis" was suggested as wider in scope and more powerful than logic. Semantic Analysis in this sense is closely akin to Korzybski's Non-Aristotelian Semantics, but basically different from Chwistek's Semantik. The recent trend towards the reconstruction of logic in the work of Russell, Hilbert, Brouwer, Chwistek, Lukasiewicz, Tarski, Lesniewski and Korzybski was briefly sketched.

4. In testing for the maximum or minimum values of the radius of curvature of the parabola, $y^2=4ax$, differentiation with respect to x fails to indicate the origin as a critical point while differentiation with respect to y does indicate this point. In the case of the ellipse in standard form, differentiation with respect to x indicates one pair of points while differentiation with respect to y indicates another pair. Similar results are found for the hyperbola. The aim of this paper is to point out the apparent exceptions to ordinary rules and to indicate the reasons for these conditions.

5. In this paper models were presented showing families of solids formed by the combinations of the regular convex polyhedrons. Professor Heath presented to the members of the Association pamphlets of patterns for the construction of twelve of these solids, as, for example, the formation of a solid composed of equal octohedrons mounted on each face of an icosahedron.

6. In the analysis of series of economic items, as, for example, the Dow Jones stock market averages, pig iron production, etc., it is desirable to have a technique for fitting trend lines of higher degree than the straight line and also to be able to compute the correlations of the deviations of these series from their trend lines. By means of simple formulas involving quadratic forms of the moments of the series it is possible to solve this problem in the sense of least squares. Explicit formulas for the coefficients of the forms in terms of the number of items and tables of their values are given for polynomials from the first to the seventh degrees inclusive. The speaker indicated the application of these formulas in an elaborate computation of economic constants which is being made by the Cowles Commission for Research in Economics of Colorado Springs, an affiliate of the Econometric Society.

7. In this paper Professor Dotterer presented the case of higher geometry in the college curriculum. He indicated the scope of this subject which seems desirable for presentation to undergraduates and pointed out the many contacts which this study makes with other mathematical disciplines.

8. In this paper Professor Edington discussed the results obtained by permuting the coefficients of such equations as $y=ax^2+bx+c$, $x^2+y^2+ax+by+c=0$, $ax+by+c=0$, and indicated the generalizations and some of the geometrical relations associated with group concepts. It was also pointed out that Veronese in 1881 had discovered that the permutations of the homogeneous coordinates of a point in a plane gave six points which are the vertices of a Pascal hexagon, and Professor Edington showed that the plane is divided up into regions such that if a point is taken from a given region, the kind of conic determined by the six points is always of the same kind. The extension to space was also indicated.

Resolutions were adopted by the members of the section expressing their appreciation and thanks to the authorities of Butler University for their hospitality, to Professor Bliss, Professor Pollock, and Professor Lanczos for their contributions to the program, and to the Mathematics Section of the State Teachers' Association and the Extension Division of Indiana University for their efforts in stimulating mathematical study through the recently inaugurated state wide contest in mathematics for high school study.

H. T. DAVIS, *Secretary-Treasurer*

THE THIRTEENTH ANNUAL MEETING OF THE ILLINOIS SECTION

The thirteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, on Friday and Saturday, May 6 and 7, 1932.

The attendance was about ninety, including the following forty-five members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. W. Bailey,

G. A. Baker, R. W. Barnard, Walter Bartky, H. R. Beveridge, Julia W. Bower, O. K. Bower, R. D. Carmichael, A. B. Coble, C. E. Comstock, A. R. Crathorne, H. B. Curry, D. R. Curtiss, Arnold Emch, Elinor B. Flagg, H. L. Garabedian, A. E. Gault, R. M. Ginnings, H. W. Haggard, Mildred Hunt, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, Harry Levy, Mrs. Mayme I. Logsdon, E. B. Lytle, W. D. MacMillan, E. J. McShane, H. J. Miles, G. A. Miller, C. N. Mills, G. E. Moore, E. J. Moulton, J. W. Peters, W. T. Reid, C. K. Robbins, H. A. Simmons, H. E. Slaught, C. J. Stowell, E. H. Taylor, V. B. Teach, E. J. Townsend.

The Section chairman, Professor R. W. Barnard, presided at both the Friday afternoon and Saturday morning sessions.

At the business session officers for 1932–1933 were elected as follows: Chairman, W. C. Krathwohl, Armour Institute of Technology; vice-Chairman, E. B. Miller, Illinois College; Secretary-Treasurer, C. N. Mills, Normal University; Assistant Secretary-Treasurer, Edith I. Atkin, Normal University.

The program consisted of eight papers, as follows:

1. "An elementary development of Riemann's geometry with applications to relativity" by Professor Cornelius Lanczos, Purdue University, by invitation.

2. "A method of Prüfer for differential systems of the second order" by Doctor W. T. Reid, University of Chicago.

3. "The Hamilton-Jacobi theory for problems in parametric form" by Professor V. B. Teach, Armour Institute.

4. "On mathematical models and graphs," accompanied by lantern slides and exhibit of models, by Professor Arnold Emch, University of Illinois.

5. "Recent discoveries in the history of mathematics" by Professor G. A. Miller, University of Illinois.

6. "Permanent configurations in the four-body problem" by Professor Walter Bartky, University of Chicago.

7. "A problem of Klein concerning the real roots of an algebraic equation" by Doctor I. J. Schoenberg, University of Chicago, by invitation.

8. "The transformations of non-normal frequency distributions into normal distributions" by Professor G. A. Baker, Shurtleff College.

Abstracts of some of these papers follow:

1. The author wishes to sketch how far we can come in understanding the general principles of the Riemannian geometry, and its applications to physical problems, if we utilize only elementary knowledge, and do not introduce the general tensor-calculus. After bringing in the fundamental line-element of Riemann, and defining the straight line as the shortest line, he demonstrates as a first application the parallel development of plane, spherical, and hyperbolic geometry based upon the corresponding line elements, and there follows the deduction of the centrifugal and Coriolis force, with the help of the line-element as an example for "apparent-forces" in mechanics. The equivalence

hypothesis of Einstein and its first attempt to solve the problem of gravitation can also be included in this elementary development, obtaining finally a good foundation in order to study the higher chapters of Riemannian geometry and of relativity.

2. In this paper the method introduced by Prüfer (Mathematische Annalen, Vol. 95 (1926), pp. 499–518) is applied to a system of two ordinary linear homogeneous differential equations of the first order; and properties of the solutions of such a system, including comparison and oscillation theorems, are discussed.

3. A classical formulation of the Hamilton principle is given and the Lagrange differential equations defining the motion of a dynamical system derived and expressed in canonical form. Next a brief discussion is given on the theory relating to the partial differential equation of Hamilton and Jacobi, and on the Jacobi theorem on the determination of solutions of the Lagrange equations from a complete integral of this partial differential equation. Professor Teach then shows how the Hamilton-Jacobi theory may be extended, and how analogous results may be obtained when parametric representation is used.

6. A permanent configuration is a configuration of n-particles moving under their mutual gravitational attraction in such a way that ratios of mutual distances are not changed. The only permanent configurations in the problem of three bodies are the straight line and equilateral triangle solutions of Lagrange. A generalization of the straight line configuration to n-bodies is given in F. R. Moulton's Periodic Orbits. In this paper all remaining permanent configurations in the problem of four bodies are found.

7. In a paper entitled *Geometrisches zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Collected Works, vol. 2, pp. 198–208, Felix Klein initiated the problem to compare the efficiency of the various methods (Budan-Fourier, Descartes-Jacobi, Newton-Sylvester) which give upper limits for the number of real roots of an algebraic equation within a given interval. The method used by Klein is geometrical and gives definite results in case the discussion involves the consideration of spaces of two or three dimensions only. The method of the present paper is analytical and is based on some general results given by the author in his paper *Über variationsvermindernde lineare Transformationen*, Mathematische Zeitschrift, vol. 32 (1930), pp. 321–328. Klein has shown in particular that for equations of the second degree the method of Descartes-Jacobi is always more efficient than the method of Budan-Fourier (See H. Weber, *Lehrbuch der Algebra*, vol. 1, second edition, 111, pp. 354–357). H. Weber (loc. cit. p. 357) asks if this is true for any degree. In the present paper it is first proved that for equations of any degree the method of Descartes-Jacobi is always more efficient than a method given by Laguerre (Oeuvres, vol. 1, Paris 1898, p. 10). The comparison of the former method with the method of Budan-Fourier is reduced for a given degree n to the discussion of the signs of the minors of a particular matrix of $n+1$ rows and $2n+2$ columns whose elements are binomial coefficients. For a given degree this discussion will answer Weber's question by a finite number of trials.

8. By assuming that the transforming function necessary to transform a given non-normal distribution into a normal distribution can be represented by a Maclaurin's expansion, Professor Baker shows how to determine the coefficients of the transforming function in terms of the coefficients of the expansion of the given distribution. The transformation necessary to transform a given non-normal distribution into a normal distribution serves to specify the distributions in random samples of the estimates of the parameters of the non-normal distribution in terms of the distributions in random samples of the mean and standard deviation of the resulting normal distribution. Thus it is possible to determine the distributions in random samples of the parameters of the non-normal population.

C. N. MILLS, *Secretary*

CONFORMAL REPRESENTATION, WITH APPLICATIONS TO PROBLEMS OF APPLIED MATHEMATICS[1]

By WARREN WEAVER, University of Wisconsin

I. INTRODUCTION TO CONFORMAL REPRESENTATION

The history of the development of mathematics is intimately connected with the history of the development of the number concept. One of the most significant and fruitful of the various generalizations which the number concept has experienced is that which led to "imaginary" and "complex" numbers. Since the product of any real number, positive or negative, by itself is positive, it is clear that there is no ordinary or real number whose square is negative; and hence it was natural to call "imaginary" the square root of a negative number. Since, moreover, the customary laws of algebra permit one to write, for any number a,

$$\sqrt{-a^2} = \sqrt{(-1)a^2} = \sqrt{-1}\,a,$$

it is clear that the pure imaginary unit $\sqrt{-1} \equiv i$ can, so to speak, be made to bear the entire brunt of "imaginariness," and the square root of any negative number can be written as the product of the imaginary unit and a real number.

It is often convenient, when multiplying one real number a by a second positive real number k to give, on geometrical grounds, an operational interpretation to the product ka; and say that k is an operator which stretches a in the ratio of k to 1. If the stretching operator be negative, and say equal to $-k$, one naturally says that $-k=(-1)k$ carries out two operations, the first of which is a stretch in the ratio of k to 1, while the second is a reversal of direction. This reversal of direction may itself conveniently be viewed as a rotation through 180°, so that -1 is an operator which leaves lengths unchanged but which rotates through $+180°$. (See Figure 1.)

[1] A lecture delivered by invitation at the S.P.E.E. Summer Session for Teachers of Mathematics to Engineering Students at Minneapolis, September, 1931.

If one seeks, similarly, to give an operational interpretation to the process of multiplying a real number b by $i \equiv \sqrt{-1}$, he receives a useful suggestion from the fact that a double application of the operation of multiplying by i is equivalent to the operation of multiplying by -1. It clearly is consistent with this fact to agree to treat $i \equiv \sqrt{-1}$ as an operator which rotates through half of 180° or 90°.

A mixed number $a+ib$ which consists of the sum of a real portion a and a pure imaginary portion ib is called a "complex" number, and the operational interpretation just suggested indicates the manner in which one may give a convenient geometrical interpretation to such a complex number. (See Figure 1.) When complex numbers are represented in this familiar way, the figure is called a Gauss-Argand diagram. It is clear, from such a diagram, that complex numbers add vectorially; that two complex numbers are equal if and only if their real and imaginary parts are separately equal. It is also clear that any

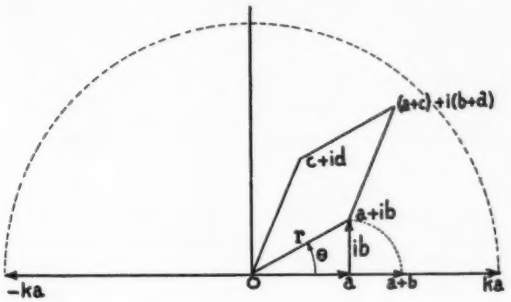

FIGURE 1

complex number $a+ib$ can be written in the equivalent form $r(\cos\theta+i\sin\theta) \equiv re^{i\theta}$. This latter expression is called the polar form of the complex number, r and θ being known as, respectively, the modulus and the amplitude of $a+ib$. The modulus, or "absolute value" of a complex number z is often represented by $|z|$.

It is readily argued, using the polar form, that the product (or quotient) of two complex numbers is a complex number whose modulus is the product (or quotient) of the moduli of the factors and whose amplitude is the sum (or difference) of the amplitudes of the factors.

A complex quantity whose real and pure imaginary parts are variable is called a complex variable, and may be denoted by $z=x+iy$ or by $w=u+iv$. A functional relationship $w=f(z)$ may exist between two such complex variables, so that to each chosen value of z there corresponds a value of w. Each of the complex quantities w and z can be represented on an Argand diagram, so that one speaks of the w-plane and the z-plane. The functional relation sets up a

correspondence between the points of the z-plane and points of the w-plane. Thus when the variable z wanders, in its plane, over a curve C the corresponding point w describes a curve C' in the w-plane. One of these curves is said to be the "map" of the other; and in this way a functional relationship between w and z maps the z-plane onto the w-plane.

If one seeks to calculate, for this situation, a quantity analogous to the derivative of a function of a real variable, significant new considerations enter. One naturally chooses, for the definition of the derivative,

$$\frac{dw}{dz} = \lim_{\Delta z \doteq 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \doteq 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

and the new feature arises from the fact that $\Delta z = \Delta x + i\Delta y$ has, geometrically speaking, two degrees of freedom and can approach zero along any path such as that shown in Figure 2, while Δx, in the simple previous case, was restricted to linear variation. In order that the derivative of w with respect to z have a

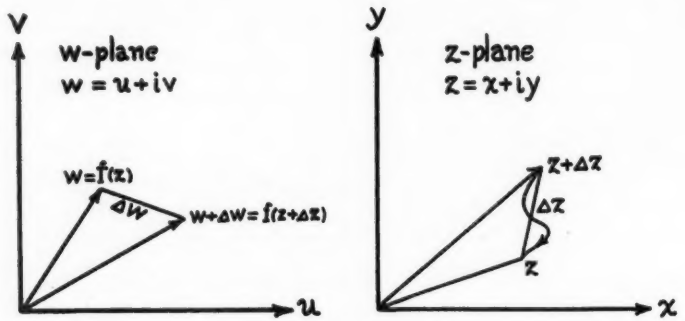

FIGURE 2

unique value for a given argument, it is necessary to demand that the limit of the above difference quotient be independent of the manner in which Δz approach zero. If one assume, for the moment, that $w = u + iv = f(z)$ have a unique derivative with respect to $z = x + iy$ one has

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz},$$

while

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = i\frac{dw}{dz}.$$

Therefore

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right),$$

from which, equating reals and pure imaginaries,

$$(1) \qquad \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

These are known as the Cauchy-Riemann equations, and it can easily be shown that a necessary and sufficient condition that a complex function $w=u+iv$ possess a derivative with respect to $z=x+iy$ is that the partial derivatives of the functions u and v exist, are continuous, and satisfy equations (1). The above calculation proves the necessity of the condition just stated. A function of a complex variable which possesses a derivative at every point of a region is said to be "analytic," "regular," or "holomorphic" over that region.

Suppose now that $w=u+iv$ is, over a certain region, an analytic function of $z=x+iy$. The Cauchy-Riemann equations (1) are then satisfied. If one differentiates the first of these equations partially with respect to x, the second partially with respect to y, and then adds the two resulting equations he obtains

$$(2) \qquad \nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This partial differential equation is known as the Laplace equation, and it is of very widespread occurrence in problems of pure and applied mathematics. This is the fundamental equation, for example, in problems of electrostatics, of magnetostatics, of steady state current flow, of the irrotational motion of a perfect non-compressible fluid, of elasticity, of sound, and of a wide variety of thermal and optical problems. By interchanging the variables when differentiating equations (1), one obtains

$$(2') \qquad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

so that *every analytic function automatically furnishes one with a pair of real functions of two real variables each of which is a solution of Laplace's equation.* The function $v(x, y)$ is called the "conjugate" of the function $u(x, y)$, and vice versa; and the process of obtaining solutions of the Laplace equation in this way from analytic functions of a complex variable is known as the "method of conjugate functions."

The actual process of obtaining the pair of conjugate functions from a given analytic function is best indicated by means of an example. Thus, if

$$w = u + iv = \sin z = \sin (x + iy),$$

then

$$\begin{aligned} u + iv &= \sin x \cos iy + \cos x \sin iy, \\ &= \sin x \cosh y + i \cos x \sinh y; \end{aligned}$$

so that

$$\begin{aligned} u(x, y) &= \sin x \cosh y, \\ v(x, y) &= \cos x \sinh y. \end{aligned}$$

In general, the relations

$$\begin{aligned} u &= u(x, y), \\ v &= v(x, y), \end{aligned} \tag{3}$$

obtained by breaking up into real and pure imaginary parts an analytic function $w=u+iv=f(z)=f(x+iy)$, can be solved for x and y to obtain the equations

$$\begin{aligned} x &= x(u, v), \\ y &= y(u, v). \end{aligned} \tag{3'}$$

These new functions x and y are also solutions of the Laplace equation, considering u and v as the independent variables. Thus, strictly speaking, each analytic function furnishes one with *four* solutions of Laplace's equation, but

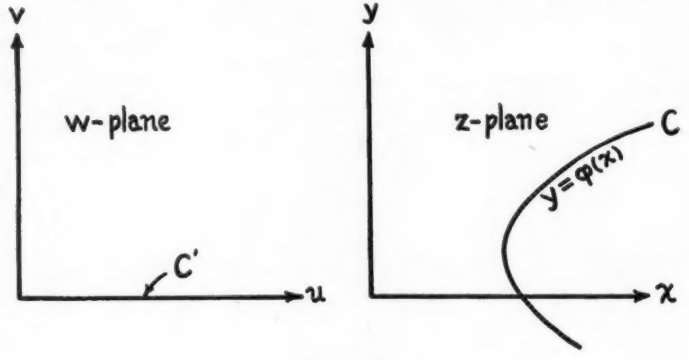

FIGURE 3

very seldom are all four sufficiently simple to be useful. The solution of (3) to produce (3') is always theoretically possible, the Cauchy-Riemann conditions for analyticity assuring that the requirements for the possibility of such solution are fulfilled.

Specific examples of the method of conjugate functions will be given later, but it may be well to indicate here two general sorts of problems. Suppose that an analytic function $w=u+iv=f(z)=f(x+iy)$ maps a curve C of the z-plane

(see Figure 3), whose equation is $y=\phi(x)$ onto the entire real axis $v=0$ of the w-plane. This will obviously occur if and only if

$$v(x, \phi(x)) \equiv 0.$$

Then the function

$$\Phi(x, y) \equiv v(x, y),$$

clearly is a solution of Laplace's equation which reduces to zero on the curve C. In an important class of problems of applied mathematics one requires a solution of Laplace's equation which reduces to zero, or some other constant, on some given curve. Thus, one may, so to speak, go at such problems backward; and by plotting in the z-plane the curves $u(x, y)=$constant and $v(x, y)=$constant, he finds for what curves C a given analytic function solves the above problem. Similarly, one may interchange the roles of u, v and x, y and may plot in the w-plane the curves $x(u, v)=$constant and $y(u, v)=$constant. Thus a properly drawn picture of the plane transformation indicates to the eye what problems, of this sort, are solved by a given analytic function. It must be emphasized that the picture must be "properly drawn"; that is, one requires, in one plane, the two families of curves obtained by setting equal to various constants the coordinate variables of the other plane.

In a second and more general sort of problem it is necessary to obtain a solution $\Phi(x, y)$ of Laplace's equation which, on a given curve C whose equation is $y=\phi(x)$, reduces to some given function $\Phi^*(x, y)$. The previous problem is clearly a very special case of this second problem. Suppose now that an analytic function $w=f(z)$ map the curve C of the z-plane onto the axis of reals $v=0$, of the w-plane. Since the curve C maps onto $v=0$ in the w-plane, $v(x, \phi(x))\equiv 0$, and the values of Φ^* at points on C are equal to the values of

$$\Phi^*(x(u, 0), y(u, 0)) \equiv \Phi_*(u)$$

at the corresponding points on the transformed curve $v=0$. Suppose now that the function $\Psi(u, v)$ be a solution of Laplace's equation (viewing u and v as independent variables), such that

$$\Psi(u, 0) \equiv \Phi_*(u).$$

It is easily checked that

$$\Phi(x, y) = \Psi(u(x, y), v(x, y))$$

is a solution of Laplace's equation, viewing x and y as independent variables. Moreover, on the curve C one has

$$\Phi(x, \phi(x)) = \Psi(u, 0) = \Phi_*(u) = \Phi^*(x, y),$$

so that Φ is the solution sought.

The chief service, in this case, of the method of conjugate functions, is that the form of the boundary condition is much simplified. Rather than seeking a

function which takes on prescribed values on some *curve* C, one has rather to find a function which takes on prescribed values on a *straight line*, namely the axis of abscissas. This latter problem is so much simpler than the former that it can, indeed, be solved in general form for a very general function Φ_*. This solution will be referred to later, in III (C).

The foregoing discussion has briefly indicated the way in which an analytic function $w=f(z)$ maps the z-plane onto the w-plane. A characteristic and important feature of this mapping has not been mentioned as yet, and is this: If one consider two curves C_1 and C_2, in the z-plane, which intersect at a point P (See Figure 4), then it may easily be shown that, if the mapping function $w=f(z)$ be analytic, the angle of intersection τ of these two curves is equal to the angle τ' at which the transformed curves C_1' and C_2' intersect, in the w-plane, at the transformed point P'. This fact can conveniently be referred to by saying that the map is "angle-true." Mathematicians, however, refer to this

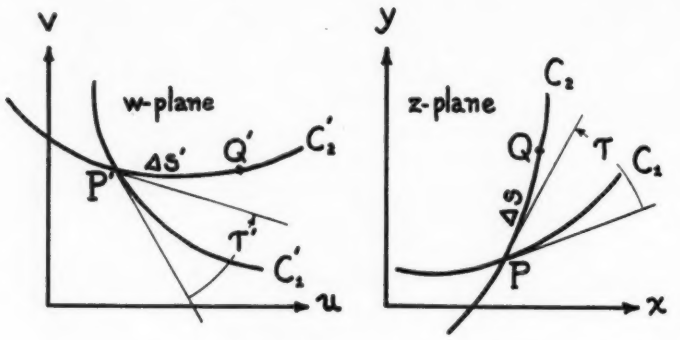

FIGURE 4

characteristic feature by saying that the map is "conformal," and one speaks of the conformal representation of one plane on the other.

As an immediate and important application of the fact that angles are preserved in the conformal map furnished by an analytic function, one may note that the curves $u=$constant of the z-plane are everywhere orthogonal to the curves $v=$constant, merely because the curves $u=$constant and $v=$constant are, in the w-plane, straight lines which intersect orthogonally. This orthogonality property is often very important in the application of conformal representation to problems of applied mathematics. If, for example, $u(x, y)$ represent the potential function in an electrical problem, then the orthogonal family of curves $v(x, y)=$constant gives the lines of force (and vice versa). Other examples of this will be mentioned later.

If one consider an increment of arc length $\Delta s=\overline{PQ}$ (Figure 4) along one of the curves in the z-plane, this transforms into $\Delta s'=\overline{P'Q'}$ in the w-plane; and it

follows at once from the uniqueness of the derivative that, regardless of the direction of the curve in question,

$$\lim \frac{\Delta s'}{\Delta s} = \left| \frac{dw}{dz} \right|.$$

Thus the region immediately around the point P of the z-plane experiences, when it is mapped on the w-plane, a uniform radial stretch whose magnitude (either greater or less than unity) is the absolute magnitude of the derivative, at that point, of w with respect to z. For this reason the quantity $|dw/dz|$ is called the "modulus of the transformation."

The above remarks have suggested the practical importance of being able to construct or determine an analytic function which will conformally map a given region of the z-plane onto some standard simple region (such as the interior of the unit circle or such as the upper half plane) of the w-plane. The fundamental theorem of conformal representation, first treated by Riemann in his doctoral dissertation, asserts that there always exist analytic functions which will accomplish this result, there being, in the simple forms of the theorem, mild restrictions on the nature of the original region of the z-plane. This fundamental theorem has been widely generalized, and has received much recent study.

II. EXAMPLES OF CONFORMAL MAPS

As a preparation for the consideration of applications, this section will present six specific instances of the conformal mapping of one plane on another. The examples chosen are not precisely those which one would select if, building up from the simplest cases, one were to study the mathematical theory in detail. The examples are chosen for their characteristic features, and because of their important and direct applications. The first case is:

(*A*) *The transformation* $w = z^m$, *m a positive integer.*

If one write both z and w in polar form, so that,

$$z = re^{i\phi},$$
$$w = Re^{i\Phi},$$

then

$$w \equiv Re^{i\Phi} = z^m = r^m e^{im\phi},$$

and

$$R = r^m,$$
$$\Phi = m\phi.$$

Thus the curve $r =$ constant in the z-plane (that is to say, a circle about the origin) transforms into a curve $R =$ constant in the w-plane (also a circle about the origin), the radius of the circle in the w-plane being equal to the mth power of the radius of the circle in the z-plane. Also, a radial line $\phi =$ constant in the z-

plane transforms into a new radial line $\Phi = \text{constant}$, the amplitude angle for the transformed radial line being m times the amplitude angle of the original radial line. Thus a sector of the z-plane of central angle $2\pi/m$ is "fanned out" to cover the entire w-plane, this sector also being stretched radially. (See Figure 5, drawn for $m = 3$.) One notes the characteristic feature that a set of orthogonal curves in one plane transform into a set of orthogonal curves in the other plane.

This example suggests several interesting questions which cannot be discussed here. The "angle-true" property clearly does not hold at the origin, which indicates that this point deserves special study. Further, it is clear that only a portion of the z-plane maps onto the entire w-plane. In the case for which the figure is drawn it would require *three* w-planes, so to speak, if the entire z-plane were to be unambiguously mapped. This consideration leads to the use of

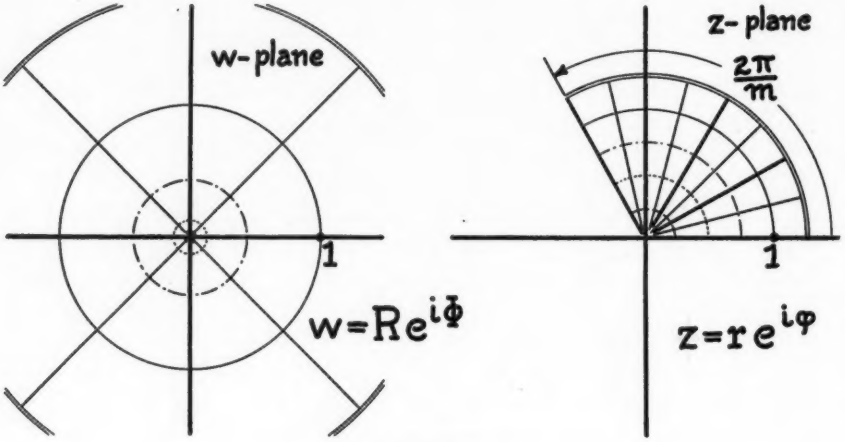

FIGURE 5
The Transformation $w = z^m (m = 3)$

many-sheeted surfaces, called Riemann surfaces. Such questions and apparent difficulties correctly indicate that a thorough knowledge of the mathematical theory of analytic functions is essential to a proper and complete understanding of even simple instances of conformal representation.[1]

To get a clear idea of the way in which the z-plane maps onto the w-plane one may choose various convenient families of curves in one plane, and determine the corresponding curves in the other plane. The resulting picture, as was mentioned earlier, does not give any indication of the immediate physical applications of the transformation in question unless one of the sets of curves, in one plane or the other, consists of the straight lines parallel to the coordinate

[1] Bieberbach, L., *Einführung in die Konforme Abbildung*, Berlin, 1927: Lewent, L., *Conformal Representation*, London, 1925: Osgood, W. F., *Lehrbuch der Funktionentheorie*, vol. 1, ch. 14.

axes. It should thus be clear that Figure 5 does not give a direct indication of the type of problem immediately solvable by the transformation $w=z^3$. The curves in the w-plane obtained by setting $x=$constant and $y=$constant are, in fact, cubic curves; and no simple physical problem is directly solved by this transformation. This transformation may, however, be used to solve various physical problems for a wedge-shaped region, since the bounding curve C of such a wedge (say the line $\phi=0$ and the line $\phi=\pi/3$) is transformed into a curve C' of the w-plane which consists of the entire real axis. Thus the transformation can be used, in the way indicated in (I), to solve problems in which one desires a solution of the Laplace equation which reduces to a given function (or a constant) on the boundary of a wedge.

(*B*) *The transformation* $w=(z+1/z)/2$.

This again is a transformation which does not have immediate applicability. It has, however, interesting features, and subsequent discussion will indicate how it may be made to serve a practical purpose.

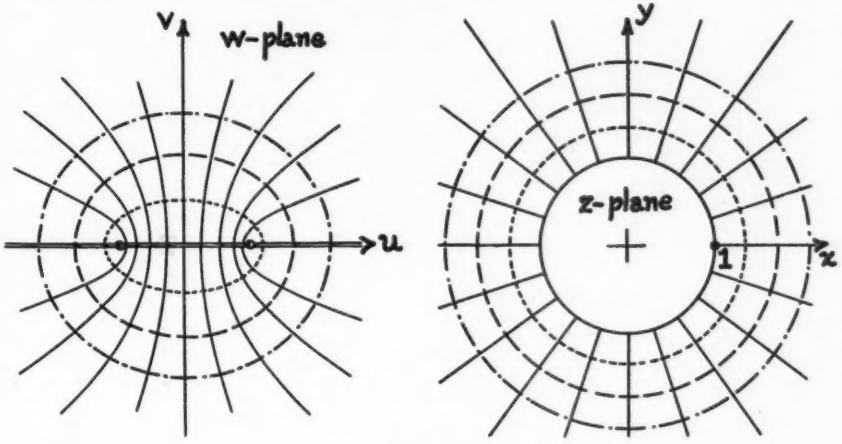

FIGURE 6
The Transformation $w=(z+1/z)/2$

If, as before, one write z in the polar form $re^{i\Phi}$, then

$$w = u + iv = (z + 1/z)/2 = re^{i\phi}/2 + e^{-i\phi}/2r,$$
$$= \tfrac{1}{2}(r + 1/r)\cos\phi + i\tfrac{1}{2}(r - 1/r)\sin\phi,$$

so that

$$u = \tfrac{1}{2}(r + 1/r)\cos\phi,$$
$$v = \tfrac{1}{2}(r - 1/r)\sin\phi.$$

Thus, eliminating ϕ and r in turn,

$$\frac{u^2}{(r+1/r)^2}+\frac{v^2}{(r-1/r)^2}=\tfrac{1}{4},$$

$$\frac{u^2}{\cos^2\phi}-\frac{v^2}{\sin^2\phi}=1.$$

From these equations it follows by inspection that the circles $r=$constant of the z-plane transform into a family of ellipses of the w-plane (See Figure 6), the ellipses being confocal since

$$(r+1/r)^2-(r-1/r)^2=4=\text{constant}.$$

It is also clear that two circles of reciprocal radii transform into the same ellipse. Similarly, the radial lines $\phi=$constant of the z-plane transform into a family of hyperbolas which, again, are confocal since

$$\cos^2\phi+\sin^2\phi=1=\text{constant}.$$

Thus the exterior of the unit circle of the z-plane transforms into the entire w-plane. The unit circle itself "flattens out" to form the segment from -1 to $+1$ of the real axis of the w-plane. All larger circles are less strenuously "flattened out" and form ellipses, while the radial lines of the z-plane form the associated confocal hyperbolas of the w-plane. A similar statement can be made for the inside of the unit circle.

(*C*) *The transformation* $w=e^z$.

If one set $w\equiv Re^{i\Phi}$ and $z=x+iy$, then

$$Re^{i\Phi}=e^{x+iy}=e^x\cdot e^{iy},$$

so that

$$R=e^x,$$

$$\Phi=y.$$

It is thus clear that vertical lines of the z-plane map into circles of the w-plane, the radius being greater or less than one, according as x is positive or negative. Horizontal lines of the z-plane, on the other hand, map into the radial lines of the w-plane, and it is clear that any horizontal strip of the z-plane of height 2π will cover the entire w-plane once. (See Figure 7.)

The curves in the w-plane of Figure 7 are drawn by setting equal to a constant one or the other of the coordinates of the z-plane. Thus these curves give direct indication of physical problems to which this analytic function may be applied. For example, one could obtain the electrostatic field due to a charged right circular cylinder, the lines of flow from a single line source of current or liquid, the circulation of a liquid around a cylindrical obstacle, etc.

By considering this example in conjunction with the preceding example, one gives new significance to Figure 6. In fact, if one start with the z-plane of Figure

7, and then use the w-plane of Figure 7 as the z-plane of Figure 6, it is clear that the curves drawn in the w-plane of Figure 6 then are obtainable by setting equal to various constants the coordinates of the z-plane of Figure 7. That is to

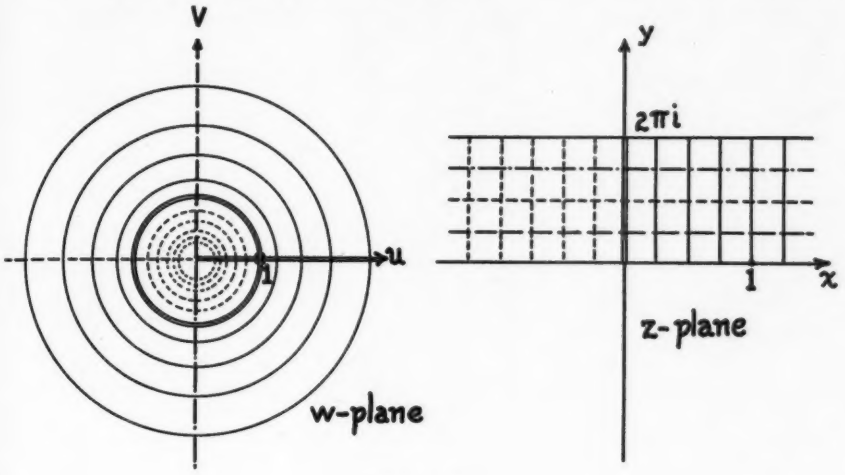

FIGURE 7
The Transformation $w=e^z$

say, the w-plane curves of Figure 6 give direct evidence of physical problems which can be solved by the pair of transformations

$$z_1 = e^z$$
$$w = (z_1 + 1/z_1)/2.$$

(*D*) *The transformation* $w = \cosh z$.

If, in the two preceding equations, one eliminate the intermediate variable z_1 (so he may pass directly from the z-plane of Figure 7 to the w-plane of Figure 6) the result is

$$w = (e^z + e^{-z})/2 = \cosh z.$$

Thus

$$u + iv = \cosh(x + iy) = \cosh x \cosh iy + \sinh x \sinh iy,$$
$$= \cosh x \cos y + i \sinh x \sin y,$$

so that

$$u = \cosh x \cos y,$$
$$v = \sinh x \sin y,$$

or

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1,$$

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1.$$

This transformation is shown in Figure 8, and it may be used to obtain the electrostatic field due to an elliptic cylinder, the electrostatic field due to a

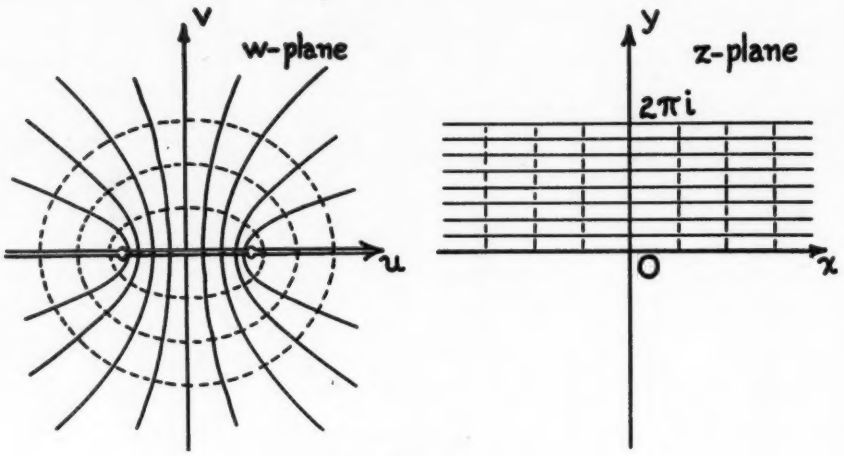

FIGURE 8
The Transformation $w = \cosh z$

charged plane from which a strip has been removed, the circulation of liquid around an elliptical cylinder, the flow of liquid through a slit in a plane, etc.

The transformation from the z-plane to the w-plane may be described geometrically as follows: Consider the horizontal strip of the z-plane between the lines $y = 0$ and $y = \pi$; and think of these lines as being broken and pivoted at the points where $x = 0$. Rotate the strip 90° counterclockwise, and at the same time fold each of the broken lines $y = 0$ and $y = \pi$ back on itself, the strip thus being doubly "fanned out" so as to cover the entire w-plane.

(*E*) *The transformation* $w = z + e^z$.

One has

$$\begin{aligned} u + iv &= x + iy + e^{x+iy}, \\ &= x + iy + e^x(\cos y + i \sin y), \end{aligned}$$

so that

$$\begin{aligned} u &= x + e^x \cos y, \\ v &= y + e^x \sin y. \end{aligned}$$

This transformation is shown in Figure 9. If one considers the portion of the z-plane between the lines $y = \pm\pi$, then the portion of the strip to the right of $x = -1$ is to be "fanned out" by rotating the portion of $y = +1$ (to the right of $x = -1$) counterclockwise and the portion of $y = -1$ (to the right of $x = -1$)

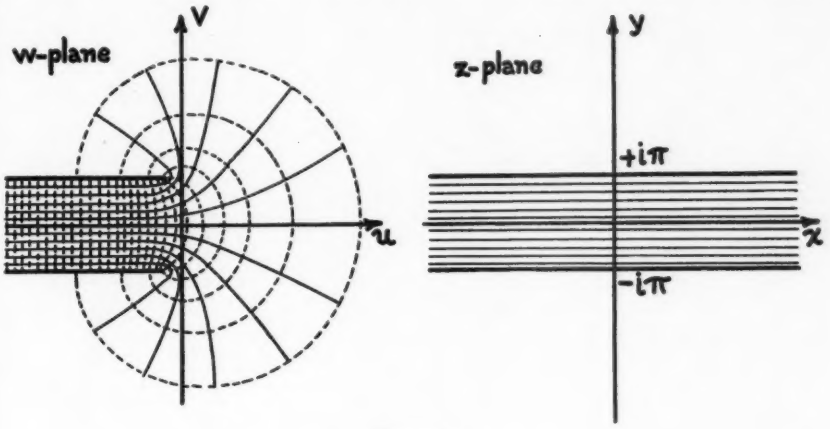

FIGURE 9
The Transformation $w = z + e^z$

clockwise until each line is folded back on itself. This transformation gives the electrostatic field at the edge of a parallel plate condenser, the flow of liquid out of a channel into an open sea, etc.

(*F*) *The Schwartz transformation.*

The transformations just considered are simple examples, and are necessarily very special in character. This list of illustrations will be concluded by a more general transformation. Suppose one have (See Figure 10) a rectilinear polygon, in the w-plane, whose sides change direction by an angle $\alpha_i\pi$ when one passes the ith vertex, going around the boundary of the polygon so that the interior lies to the left. The interior of this polygon can be mapped onto the upper half z-plane by the transformation

$$w = A \int \frac{dz}{(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_n)^{\alpha_n}} + B,$$

where $z_1, z_2, \cdots, z_n$ are the (real) points, on the x axis of the z-plane, onto which map the first, second, $\cdots$, nth vertex of the polygon; and where A and B are constants which are to be determined to fit the scale and location of the polygon. Three of the points z_i may be chosen at will, and the values of the remaining ones may be calculated.

This theorem may be used to find, for example, the analytic transformation

which solves the problem of determining the electrostatic field around a charged cylindrical conductor of any polygonal cross-section. It should be noted, however, that one requires for this purpose the function $y(u, v)$ whereas the theorem gives one w as a function of z. It is often exceedingly difficult and laborious to solve this relation for z as a function of w, so that one may obtain the function

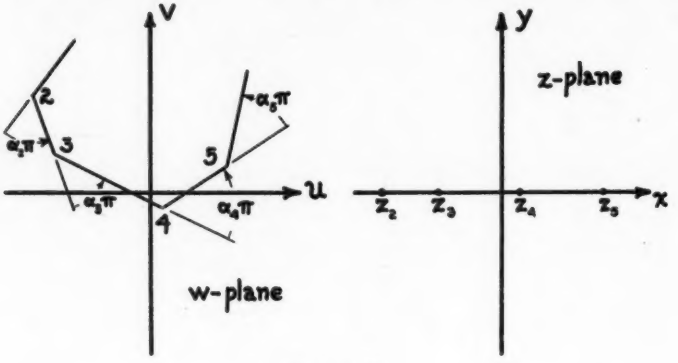

FIGURE 10

y. It should further be remarked that this theorem may be applied to polygons some of whose vertices are not located in the finite plane; and that the theorem is of wide applicability and importance in connections less direct and simple than the one just mentioned.

III. APPLICATIONS OF CONFORMAL REPRESENTATION

(*A*) *Applications to cartography.*

It is natural that a mathematical theory which discusses the "mapping" of one plane on another should have application to the problems connected with the drawing of geographic maps. Since the surface of a sphere cannot be made plane without distortion of some sort, one has to decide, when mapping a portion of the sphere on a plane, what type of distortion to choose and what to avoid. For some purposes it is essential that areas be represented properly; for other purposes it is most important that the angles on the map faithfully represent the actual angles on the sphere.

The first problem, in conveniently mapping a sphere on a plane, is to map the sphere on the plane in some fashion or other, and then, if this fashion be unsatisfactory, to remap this plane onto a second plane. The first problem can be done in a wide variety of ways[1] which includes as important examples, stereographic projection and Mercator's projection. Both of these examples are *conformal* projections, in that they preserve the true values of all angles. Having once mapped the sphere on the plane (or on a portion of the plane) one may now

[1] The Encyclopedia Britannica article on Maps lists and discusses nearly thirty such projections actually used in map making.

remap onto a second plane, and it is here that the theory of conformal representation finds its application; for one can determine the analytic function which will conformally remap the original map onto a new region of any desired shape and size. Not only are all angles preserved in this process of conformal remapping, but the distortion in the neighborhood of a point is always a pure magnification. Thus the shapes of all small objects or regions are preserved. Such maps do not give a true representation of areas, and for this reason many maps are based on compromises between conformal transformations and area-preserving transformations.

(*B*) *Applications to hydrodynamics.*

When the velocities of all particles of a moving liquid lie in planes parallel to one plane which we may conveniently choose as the x, y plane, and when all particles having the same x and y have equal velocities; then the motion is said to be two-dimensional. Such cases clearly arise if a very thin sheet of liquid is flowing in some manner over a plane; or if a thick layer of liquid circulates over a plane, there being no motion and no variation of motion normal to the plane. Let the x and y components of velocity at any point (x, y) be u and v respectively. The motion is said to be *irrotational* if the curl of the velocity vector vanishes. Analytically this demands that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

whereas physically it states that the angular velocity of an infinitesimal portion of the liquid is zero. The equation just written assures that

$$-(u\,dx + v\,dy)$$

is the perfect differential of some function, say Φ. This function is known as the velocity potential, since by a comparison of the two equations

$$d\Phi = -u\,dx - v\,dy,$$

$$d\Phi = \frac{\partial \Phi}{\partial x}\,dx + \frac{\partial \Phi}{\partial y}\,dy,$$

it follows that

$$u = -\frac{\partial \Phi}{\partial x}, \quad v = -\frac{\partial \Phi}{\partial y}.$$

Now if the liquid be incompressible the amount of it which flows into any volume in a given time must equal the amount which flows out. This demand imposes on the components of velocity the restriction that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

this being known as the "equation of continuity." From the last two equations it follows that

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\equiv \nabla^2\Phi = 0.$$

That is, the velocity potential satisfies the Laplace equation.

Just as the vanishing of the curl of the velocity demands that $u\,dx+v\,dy$ be an exact differential, so the equation of continuity demands that $v\,dx-u\,dy$ be an exact differential of some function, say ψ. That is,

$$d\Psi = v\,dx - u\,dy,$$

$$d\Psi = \frac{\partial\Psi}{\partial x}\,dx + \frac{\partial\Psi}{\partial y}\,dy,$$

so that

$$v = \frac{\partial\Psi}{\partial x}, \quad u = -\frac{\partial\Psi}{\partial y}. \tag{5}$$

From (4) and (5) it follows at once that

$$\frac{\partial\Phi}{\partial x}\frac{\partial\Psi}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial\Psi}{\partial y} = 0,$$

which expresses the geometrical fact that the curves $\Phi=$constant and $\psi=$constant intersect everywhere orthogonally. It is clear from (4) that there is no component of velocity in the direction of the curves on which Φ is a constant, so that the velocity of the liquid is everywhere orthogonal to the equi-potential curves $\Phi=$constant. That is, the curves $\psi=$constant depict everywhere the direction of flow. For this reason ψ is called the stream function and the curves $\psi=$constant are called the stream lines. From (5) and the vanishing of the curl of the velocity it follows that the stream function ψ is also a solution of the Laplace equation.

Thus the velocity potential Φ and the stream function Ψ in the case of the irrotational flow of a perfect incompressible liquid both satisfy the Laplace equation; while the curves $\Phi=$constant and $\psi=$constant form two orthogonal families. Every analytic function therefore furnishes the solution to four such problems, the four solutions resulting from the fact that one may choose the pair x, y or the pair u, v as independent variables, and that one may interchange the roles of the potential function and the stream function. Figure 8, for example, indicates two of the four problems solved by the analytic transformation $w=\cosh z$. If one treat u and v as the independent variables, and identifies the (solid) curves $y(u, v)=$constant in the w-plane with the curves $\Phi=$constant, then the dotted curves $x(u, v)=\psi=$constant give the stream lines, and one has

solved the problem of the circulation of liquid around an elliptic cylinder. If, however, one set $y(u, v) = \psi$ and $x(u, v) = \Phi$, then the solid curves of the w-plane are the stream lines, and one has solved the problem of the flow of liquid through a slit. The other two problems solved by this same function are to be obtained by drawing, in the z-plane, the curves $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$, and identifying ψ and Φ with u and v (and vice versa). The z-plane curves $u = \text{constant}$ and $v = \text{constant}$ are very complicated and do not correspond to any simple or important physical problem, and hence they are not drawn on the figure. In fact, it is usually the case that only two of the possible four problems are sufficiently simple to be of any practical use.

It should be emphasized that it is never sufficient, in obtaining the analytical solution of a definite physical problem, merely to know that certain functions satisfy the Laplace equation. One must also have certain boundary conditions. The graphs shown above disclose to the eye what physical problem has been solved precisely because they show what sort of boundary conditions are satisfied. For example, if the dotted curves of Fig. 8 are stream lines, then the problem solved is the circulation around an elliptical obstacle just because these dotted stream lines satisfy the boundary condition for such a problem; namely, because the flow at any point on the boundary of the obstacle is parallel to the boundary of the obstacle.

It is interesting to note that this same transformation $w = \cosh z$ (or, slightly more generally, $w = a \cosh z$) can be used to solve a hydrodynamic problem of a different sort. When liquid seeps through a porous soil, it is found that the component in any direction of the velocity of the liquid is proportional to the negative pressure gradient in that same direction. Thus, in a problem of two dimensional flow,

$$u = -k\frac{\partial p}{\partial x}, \quad v = -k\frac{\partial p}{\partial y}.$$

If these values be inserted in the equation of continuity, namely, in the equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

the result is

$$\nabla^2 p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$

Suppose, then, one consider the problem of the seepage flow under a gravity dam which rests on material which permits such seepage. One seeks (see Figure 11) a function p which satisfies the Laplace equation, and which satisfies certain boundary conditions on the surface of the ground. Namely, the pressure must be uniform on the surface of the ground upstream from the heel of the dam, and must be zero on the surface of the ground downstream from the toe of the dam.

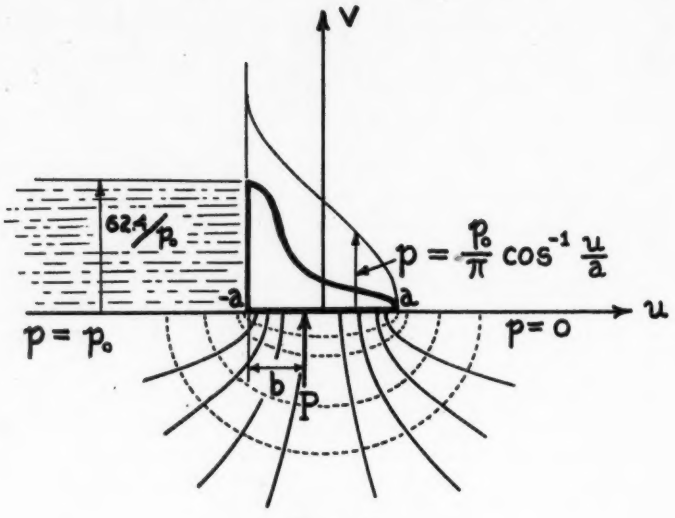

FIGURE 11
Seepage under a dam

If we choose a system of Cartesian coordinates u, v with origin at the mid-point of the base of the dam (Figure 11) and u-axis on the surface of the ground, then it is easily checked that $p(u, v) = p_0 y(u, v)/\pi$, where

$$w = u + iv = a \cosh (x + iy),$$

satisfies the demands of the problem. In fact, it was seen in II(D) where the transformation $w = \cosh z$ was studied, that the line $y = \pi$ of the z-plane folds up to produce the portion to the left of $u = -1$ of the u-axis in the w-plane; while the line $y = 0$ of the z-plane folds up to produce the portion to the right of $u = +1$ of the u-axis. The introduction of the factor a in the transformation merely makes the width of the base of the dam $2a$ rather than 2. These remarks show that $p(u, v)$ reduces to the constant π on the surface of the ground upstream from the heel of the dam. If the head above the dam is such as to produce a hydrostatic pressure p_0, one merely has to set

$$p(u, v) = p_0 y(u, v)/\pi.$$

One may now easily find the distribution of uplift pressure across the base of the dam. In fact, the base of the dam is the representation, in the u, v plane, of the line $x = 0, 0 \leqq y \leqq \pi$ of the x, y plane. Hence on the base of the dam the equations

$$u = a \cosh x \cos y,$$

$$v = a \sinh x \sin y,$$

reduce to

$$u = a \cos y,$$
$$v = 0,$$

so that

$$p(u, 0) = \frac{p_0}{\pi} \cos^{-1} \frac{u}{a}.$$

This curve is drawn in the figure. The total uplift pressure (per foot of dam)

$$P = \frac{p_0}{\pi} \int_{-a}^{+a} \cos^{-1} \frac{u}{a} \, du = p_0 a$$

which is what the uplift pressure would be if the entire base of the dam were subjected to a head just one-half the head above the dam; or if the pressure decreased uniformly (linearly) from the static head p_0 at the heel to the value zero at the toe. The point of application of the resultant uplift is easily calculated to be at a distance $b = 3a/4$ from the heel of the dam.

(*C*) *Applications to elasticity.*

If opposing couples be applied to the ends of a right cylinder or prism of homogeneous material the cylinder twists and shearing stresses are developed. Choose the axis of the prism for the z-axis of a rectangular system of coordinates. The angle of twist per unit length, say τ, and the shearing stresses, due to an applied couple T, can both be calculated if one can determine a function $\Phi(x, y)$ satisfying the Laplace equation and reducing, on the boundary of a section of the prism, to the function $\Phi^* = (x^2+y^2)/2$. In fact[1]

$$\tau = T/C,$$

where

$$C = 2G \iint (\Phi - \Phi^*) dx \, dy,$$

in which G is the modulus of rigidity of the material; whereas the shearing stresses are given by

$$X_z = G\tau\left(\frac{\partial \Phi}{\partial y} - y\right),$$
$$Y_z = -G\tau\left(\frac{\partial \Phi}{\partial x} - x\right).$$

Exact analytical solutions of this important technical problem have been obtained for several simple sections, notably circular, elliptical, rectangular, and

[1] Love, A. E. H., *Theory of Elasticity*, 3d edition, 1920, pp. 315–333.

triangular.[1] Only very recently[2] the problem was solved for an infinite T section (See Figure 12). From the general discussion given in Section (I) of this paper it is clear that, to solve this latter problem, one requires first an analytic function which will map the boundary of this T section onto the entire real axis of

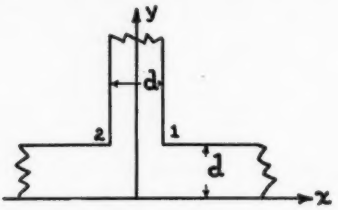

FIGURE 12
Infinite T section

the new w-plane. This section, moreover, is a rectilinear polygon, so that one can use the Schwarz transformation theory to produce the desired analytic relation. One finds that the desired mapping is carried out by the function

$$z = A \int \frac{(w^2 - 1)^{1/2}}{(w^2 - a^2)}\, dw + B,$$

$$= \frac{di}{\pi} \log \left[w + \sqrt{w^2 - 1} + \frac{2di}{\pi} \tan^{-1} \frac{\sqrt{w^2 - 1}}{2w} + \frac{d}{2} + id,\right.$$

where the first line is furnished directly by the Schwartz Theorem, and where, in the second line, the constants a, A, B have been evaluated so as to fit the dimensions and location of the T section.

It is next necessary to break z up into its real and imaginary parts so as to obtain x and y as functions of u and v. These values, when substituted into

$$\Phi^* = (x^2 + y^2)/2$$

give, since $v = 0$ on the boundary of the section, the function

$$\Phi^*[x(u, 0), y(u, 0)] \equiv \Phi_*(u).$$

The remaining essential step is to obtain a function $\Psi(u, v)$ satisfying the La-

[1] See *The torsion of members having sections common in aircraft construction*, by W. Trayer and H. W. March, Bureau of Aeronautics Navy Department, Separate Report No. 334; also contained in the 15th Annual Report of the National Advisory Committee for Aeronautics, 1929, pp. 675–719.

[2] *On a solution of Laplace's equation with an application to the torsion problem for a polygon with reentrant angles*, I. S. Sokolnikoff, Transactions of the American Mathematical Society, vol. 33, pp. 719–732.

place equation and reducing, on the axis of reals $v=0$, to the function $\Phi_*(u)$. Such a function is[1]

$$\Psi = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\Phi_*(\xi)\rho \sin\theta}{\rho^2 - 2\xi\rho\cos\theta + \xi^2}\, d\xi,$$

where

$$w = u + iv = \rho e^{i\theta}.$$

The solution to the original problem is then given, as was earlier indicated in Section I, by $\Phi=\Psi$. It is, obviously, a difficult and laborious job to carry out these calculations, but formulas have been obtained, in the paper referred to, from which practical calculations can be made.

(*D*) *Applications to electrostatics.*

The methods of complex variable theory are peculiarly applicable to two-dimensional electrical problems. In order that the problems be two-dimensional we will understand that the conductors under consideration are exceedingly long cylinders whose axes are normal to the $z=x+iy$ plane. Under these circumstances the various electrical quantities do not change appreciably in the direction normal to the z-plane, and one has to determine these quantities as functions of x and y only. In certain problems one or more of the conductors present will have very small cross-sections, and will be given a charge of, say, e' per unit length. Such a conductor will be called a "line charge of strength e'."

The electrostatic problem, for such conductors, is solved when one has obtained a function $\Phi(x, y)$, known as the electrostatic potential, satisfying the following conditions:[2]

(6)
- (*a*) $\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0$ at all points in free space.
- (*b*) Φ reduces, on the surface of the kth conductor, to a constant Φ_k.
- (*c*) In the neighborhood of a line charge of strength e', Φ becomes infinite as $(-e'\log r)/2\pi$, where r measures distance to the line.
- (*d*) Φ behaves at infinity as $(-\log R\Sigma e')/2\pi$, where $\Sigma e'$ is the total charge per unit length of all conductors present, and where R measures distance from some reference point in the finite plane. In case $\Sigma e' = 0$, Φ approaches zero as $1/R$.

It is readily shown, by standard methods, that the solution of such a problem is unique. This remark is of great practical importance, since it assures one that a function Φ satisfying these conditions is, however it may have been obtained, the correct solution of the physical problem.

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[1] Sokolnikoff, l.c. This formula is the general solution, spoken of in Section I, of the Laplace equation subject to specified boundary values on the entire axis of abscissas.

[2] See Mason and Weaver, *The Electromagnetic Field*, University of Chicago Press, 1929, pp. 134, 146, and Riemann-Weber, *Die Differentialgleichungen der Physik*, Vieweg & Sohn, 1927, Vol. II, p. 290. The units used in the above discussion are the rational units used in Mason and Weaver, l.c.

Physically one wishes to know the distribution of charge and the electrostatic force at any point. These data may be obtained from the function Φ in the following manner: the component E_s in any direction s of the electrostatic force per unit charge is given in terms of Φ by the relation

$$E_s = -\frac{\partial \Phi}{\partial s};$$

while the surface density of charge η on any conductor is given by

$$\eta = -\frac{\partial \Phi}{\partial n},$$

where n measures distance along the external normal to the conductor in question.

Now if

$$w = u + iv = f(z) = f(x + iy),$$

and if the function

$$\Phi(x, y) = u(x, y)$$

satisfies condition (6), then

$$E_x = -\frac{\partial \Phi}{\partial x} = -\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y},$$

the last step following from the Cauchy-Riemann equations (1) of Section I. Similarly

$$E_y = -\frac{\partial \Phi}{\partial y} = -\frac{\partial u}{\partial y} = +\frac{\partial v}{\partial x}.$$

Thus

$$E_x - iE_y = -\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} = i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right),$$

$$= i\frac{\partial w}{\partial x} = i\frac{dw}{dz},$$

the last step resulting from the fundamental fact that the value of the derivative of an analytic function w is independent of the mode in which z approaches zero. Now the complex number $a-ib$ is called the "conjugate" of the complex number $a+ib$ and one often denotes a conjugate by a bar, thus:

$$a - ib = \overline{a + b\,i}.$$

Using this standard notation, the "complex electric force" $\mathcal{E} \equiv E_x + iE_y$ is given by

$$\mathcal{E} \equiv E_x + iE_y = i\overline{\frac{dw}{dz}} = -\,\overline{i\frac{dw}{dz}}, \tag{7}$$

and the magnitude $\sqrt{E_x^2+E_y^2}$ of the electrostatic force at any point is given by

$$\sqrt{E_x^2 + E_y^2} = \left|\frac{dw}{dz}\right|. \tag{8}$$

If one chooses $\Phi \equiv v(x, y)$, then (7) and (8) are replaced by

$$\mathcal{E} = E_x + iE_y = \overline{\frac{dw}{dz}}, \tag{7'}$$

$$\sqrt{E_x^2 + E_y^2} = \left|\frac{dw}{dz}\right|. \tag{8'}$$

Three types of electrostatic problems will now be briefly considered. The first and simplest two dimensional electrostatic problem is that of a single long cylindrical conductor with a given charge. One then seeks a function which satisfies the equation of Laplace and which, in accordance with (6)*b*, reduces to a constant on the curve which bounds a section of the conductor. This is the analytical problem whose solution was indicated in Section I. One requires a function $w=u+iv=f(z)=f(x+iy)$ such that either a vertical straight line u = constant or a horizontal straight line v = constant of the w-plane maps into the bounding curve C of the conductor's section in the z-plane. Then $\Phi(x, y) \equiv u(x, y)$ or $\Phi(x, y) \equiv v(x, y)$ solves the problem, and the physically important quantities are given by (7), (8) or by (7'), (8') respectively.

Secondly, suppose that a single long cylindrical conductor is in the presence of a parallel line charge of strength e'. We suppose the line charge to be outside the conductor. Let C be the bounding curve in the z-plane of a section of the conductor, and let the line charge be located at $z=z_0$. We may conveniently suppose the cylindrical conductor to be grounded, so that we seek a solution of the Laplace equation which reduces to zero on C and which becomes infinite as indicated in (6*c*) at $z=z_0$. Let $\zeta=f(z)$ transform C onto the entire axis of reals and the exterior of C conformally upon the upper half ζ-plane. Then if

$$w = u + iv = \frac{e'}{2\pi} \log \frac{f(z) - \overline{f(z_0)}}{f(z) - f(z_0)},$$

the function $\Phi \equiv u(x, y)$ is the solution sought. In fact, for values of z sufficiently close to z_0, $f(z)-f(z_0)$ behaves, except for a constant factor, as $(z-z_0)$. Thus, if one write

$$z - z_0 = re'^{i\theta},$$

then for values of z very near to z_0,

$$w = u + iv = \frac{e'}{2\pi} \log \frac{1}{re^{i\theta}} + A = \frac{e'}{2\pi} \log \frac{1}{r} - i\frac{e'\theta}{2\pi} + A,$$

where A remains finite as $z \doteq z_0$. Therefore

$$u = \frac{e'}{2\pi} \log \frac{1}{r} + B,$$

where B remains finite as $z \doteq z_0$. Thus $u(x, y)$ has the proper type of infinity at $z = z_0$. Furthermore, for points z on C, $f(z)$ is on the axis of reals in the ζ-plane, so that the modulus of $f(z) - f(z_0)$ equals the modulus of $f(z) - \overline{f(z_0)}$. Hence the modulus of

$$\frac{f(z) - \overline{f(z_0)}}{f(z) - f(z_0)}$$

is unity. However, since

$$\log \rho e^{i\phi} = \log \rho + i\phi$$

it is clear that the real part of the logarithm of a complex quantity is the logarithm of the modulus of the complex quantity. Since the logarithm of unity is zero it is clear that u vanishes on C. As regards the behavior of $u(x, y)$ at infinity, one notes that u is the logarithm of the ratio of the (real) distances of $\zeta = f(z)$ to $\zeta_0 = \overline{f(z_0)}$ and to $\zeta_0 = f(z_0)$. As z becomes infinite this ratio differs from

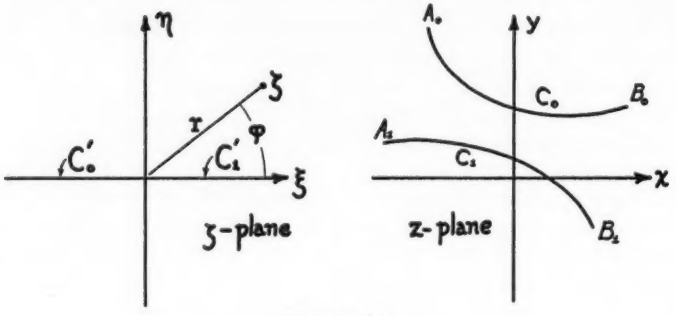


FIGURE 13

unity by an amount whose leading term is equal to or less than a constant times the reciprocal distance from $f(z)$ to one of the points $f(z_0)$ or $\overline{f(z_0)}$. Thus the leading term in the logarithm of this ratio is a constant times this reciprocal distance; and $\Phi = u$ behaves at ∞ in the required manner.

In the third type of problem there are two conductors present, one raised to the potential Φ_0 while the other is at a potential zero. Thus suppose that the cross-section of two long cylindrical conductors consists of two curves C_0 and C_1, such as those shown in Figure 13, which do not intersect at a finite point, but

which, if one takes account of the intersection of B_1 and B_0 and of A_0 and A_1 at $z=\infty$, divide the extended plane into two simply connected regions, one of which may be called the "interior" and the other the "exterior" of the closed curve C_0+C_1. Now suppose that $\zeta=f(z)$ map C_0 onto the entire negative axis of reals in the ζ-plane, with the infinitely distant point along B_0 mapped onto $\zeta=0$; and that $\zeta=f(z)$ also maps C_1 onto the entire positive axis of reals with the infinitely distant point along B_1, mapped onto $\zeta=0$; and that $\zeta=f(z)$ map the interior of C_0+C_1 conformally on the upper half ζ-plane. Then if

$$w = u + iv = \frac{\Phi_0}{\pi} \log f(z),$$

the function $\Phi=v(x, y)$ satisfies $\nabla^2\Phi=0$ at every point in the interior of C_0+C_1; reduces to zero on C_1; and reduces to Φ_0 on C_0. In fact, the imaginary part of the logarithm of a complex number is merely the amplitude of the complex number, and for points on C_1, $f(z)$ has an amplitude of π; while for points on C_0, $f(z)$ has an amplitude of zero. Then

$$\frac{dw}{dz} = \frac{\Phi_0}{\pi} \frac{ds/dz}{f(z)}$$

and the electrostatic force is given by (8) and (8′).

This third type of problem is of frequent and important practical occurrence. Many electrical engineering problems which have been solved by this method of conformal representation are referred to in an expository article, devoted largely to the Schwarz transformation, by E. Weber.[1] In an earlier article in the same journal,[2] for instance, the theory of conformal representation is applied to the problem of studying the leakage voltage and the breakdown potential between the high and low potential portions of oil-immersed transformers. The cases studied come under the third type of problem discussed above.

THE SUMMATION OF CERTAIN SERIES BY FOURIER EXPANSIONS

By JOSEPH B. REYNOLDS, Lehigh University

It is the purpose of this paper to obtain, in a simple direct manner, a method of evaluating the sum of the reciprocals of the even powers of the odd integers and the sum of the reciprocals, with alternating signs, of the odd powers of the odd integers.[3] The procedure is to develop a Fourier series, sub-

[1] E. Weber, Archiv für Elektrotechnik, 18–18, 1926–27, p. 174.

[2] L. Dreyfus, Archiv für Elektrotechnik, 13, 1924, p. 123.

[3] These sums are given, for example, in Knopp's *Theory and Applications of Infinite Series*, English Translation, page 238.

stitute in it a particular value of the variable and thus determine a relation among the sums under discussion. For the even powers the Fourier series is made to fit the curve $y=x^p$ (p a positive odd integer) from $x=0$ to $x=\pi$ and $y=(\pi-x)^p$ from $x=\frac{1}{2}\pi$ to $x=\pi$. For the odd powers the series is made to fit $y=x^p$ from $x=0$ to $x=\pi$. In both cases the particular value of x used is $\frac{1}{2}\pi$.

From the Fourier series

$$(1) \qquad f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots + a_m \sin mx + \cdots,$$

we have for $x=\frac{1}{2}\pi$

$$(2) \qquad f(\tfrac{1}{2}\pi) = a_1 - a_3 + a_5 - \cdots + (-1)^{1/2(m-1)} a_m + \cdots,$$

in which m is a positive odd integer.

If the series (1) is made to fit $y=x^p$ from $x=0$ to $x=\frac{1}{2}\pi$ and $y=(\pi-x)^p$ from $x=\frac{1}{2}\pi$ to $x=\pi$ and p is a positive odd integer

$$\begin{aligned}\tfrac{1}{2}\pi a_m &= \int_0^{1/2\pi} x^p \sin mx dx + \int_{1/2\pi}^{\pi} (\pi - x)^p \sin mx dx \\ &= (-1)^{m+1}\left[\frac{p\pi^{p-1}}{2^{p-2}m^2} - \frac{p(p-1)(p-2)\pi^{p-3}}{2^{p-4}m^4}\right. \\ &\qquad \left. + \frac{p(p-1)(p-2)(p-3)(p-4)\pi^{p-5}}{2^{p-6}m^6} - \cdots\right]\end{aligned}$$

Since m is a positive odd integer these values of a_m in (2) give

$$\begin{aligned}\frac{\pi^{p+1}}{2^{p+1}} &= \frac{p\pi^{p-1}}{2^{p-2}} S_2 - \frac{p(p-1)(p-2)\pi^{p-3}}{2^{p-4}} S_4 \\ &\quad + \frac{p(p-1)(p-2)(p-3)(p-4)\pi^{p-5}}{2^{p-6}} S_6 - \cdots \\ &\quad + (-1)^{1/2(n-2)} \frac{p!\pi^{p-n+1}}{(p-n+1)!2^{p-n}} S_n + \cdots\end{aligned}$$

in which

$$S_n = \sum_{m=1,3,\cdots}^{\infty} \frac{1}{m^n}$$

where n is any positive even integer. From this relation, by setting $p=n-1$, we derive

$$\begin{aligned}S_n = (-1)^{1/2(n+2)}\Bigg[&\frac{\pi^n}{2^{n+1}(n-1)!} - \frac{\pi^{n-2}}{2^{n-2}(n-2)!} S_2 + \frac{\pi^{n-4}}{2^{n-4}(n-4)!} S_4 - \cdots \\ &+ \frac{(-1)^{1/2n}\pi^2}{2^2(2!)} S_{n-2}\Bigg]\end{aligned}$$

For a chosen value of n, the series of terms in the bracket ends with the one having the subscript $n-2$. Thus, for $n=2$, we have

$$S_2 = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8};$$

for $n=4$ we obtain

$$S_4 = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = -\left[\frac{\pi^4}{32\cdot 6} - \frac{\pi^2}{4\cdot 2}\cdot\frac{\pi^2}{8}\right] = \frac{\pi^4}{96}$$

In this manner we may find

$$S_6 = \pi^6/960, \quad S_8 = 17\pi^8/32(7!), \quad S_{10} = 31\pi^{10}/8(9!), \quad \text{etc.}$$

It is interesting to note that from

$$S_{10} = \frac{31\pi^{10}}{8(9!)} = \frac{1}{1^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \cdots,$$

we can determine the value of π correct to six figures by using one term only of the series and the value of $\log \pi$ correct to eight places of decimals by using two terms of the series.

For the other case we make series (1) fit $y=x^p$ from $x=0$ to $x=\pi$, in which p is a positive odd integer. In this case, for m a positive odd integer, in (2)

$$\tfrac{1}{2}\pi a_m = \int_0^\pi x^p \sin mx\, dx$$

$$= \frac{\pi^p}{m} - \frac{p(p-1)\pi^{p-2}}{m^3} + \frac{p(p-1)(p-2)(p-3)\pi^{p-4}}{m^5} - \cdots,$$

and therefore,

$$\frac{\pi^{p+1}}{2^{p+1}} = \pi^p S_1 - \frac{p!\pi^{p-2}}{(p-2)!} S_3 + \frac{p!\pi^{p-4}}{(p-4)!} S_5 - \cdots + \frac{(-1)^{(n-1)/2} p!\pi^{p-n+1}}{(p-n+1)!} S_n,$$

in which

$$S_n = \sum_{m=1,3,\cdots}^{\infty} \frac{(-1)^{(m-1)/2}}{m^n},$$

where m is a positive odd integer. Replacing p by n we derive

$$S_n = (-1)^{1/2(n-1)}\left[\frac{\pi^n}{2^{n+1}n!} - \frac{\pi^{n-1}}{n!} S_1 + \frac{\pi^{n-3}}{(n-2)!} S_3 - \frac{\pi^{n-5}}{(n-4)!} S_5 + \cdots + \frac{(-1)^{(n-1)/2}\pi^2}{3!} S_{n-2}\right]$$

For $n=1$ we have the common series

$$S_1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4};$$

for $n=3$ we obtain

$$S_3 = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = -\left[\frac{\pi^3}{16(6)} - \frac{\pi^2}{6}\cdot\frac{\pi}{4}\right] = \frac{\pi^3}{32}.$$

By this formula we readily find

$$S_5 = 5\pi^5/2^6(4!),\quad S_7 = 61\pi^7/2^8(6!),\quad S_9 = 12465\pi^9/2^{10}(9!),\quad \text{etc.}$$

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE APPROXIMATE SOLUTION OF CERTAIN EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

In a recent issue of this MONTHLY[1] Mr. E. C. Kennedy discusses the approximate solution of equations of the type $x^x = c$. The present paper is in a sense a continuation of his discussion, since it supplies a theoretical justification for the method and gives applications to a number of different kinds of equations.

In the first place, Mr. Kennedy's method is really not a "new method," as the title of his note would indicate. It is simply the familiar Newton's method, applied, however, not to the given equation, but to a related equation. To show this we may as well consider a more general case. If a is an approximation to a real root of the equation $f(x) = c$, where c is a positive constant, then Newton's method gives $a + [c - f(a)]/f'(a)$ as a second approximation. In certain well-known cases this second value will be closer to the root than a. But in some instances we can do better by applying Newton's method, not to $f(x) = c$, but to $\log f(x) = \log c$. This gives as a second approximation

$$a + \frac{f(a)}{f'(a)}[\log c - \log f(a)], \tag{1}$$

which will be a better approximation than a provided the function $\log f(x) - \log c$ has suitable properties and a is properly chosen.[2] Formula (1) gives Mr. Kennedy's method when applied to $x^x = c$, as the reader can verify at once.

[1] Vol. 38, 1931, pages 449–50.

[2] We must choose a so that $f(a) > 0$.

The two second approximations obtained above can be easily compared. Suppose we write $\log c - \log f(a)$ as $\log [1 + \{c - f(a)\}/f(a)]$. This is known to be less than $[c - f(a)]/f(a)$. If then $f'(a)$ is positive, formula (1) gives a smaller value than $a + [c - f(a)]/f'(a)$, if $f'(a)$ is negative, a larger value.

It follows at once that in some cases formula (1) will give better results than Newton's method applied to the given equation. A typical case is that in which the functions $f(x) - c$ and $\log f(x) - \log c$ both have their first and second derivatives positive in the neighborhood of the root we are seeking. Since Newton's method takes us along the tangent to the curve, speaking roughly, the first approximation a should be taken larger than the root to guarantee that the second approximation is closer. We are sometimes justified, however, in taking a too small, if we can get a better first approximation by so doing and if the graph of the function is fairly steep where we are working. In such a situation the second approximation will be too large, but probably closer than the first. But no matter which sort of a first approximation we wish to use, formula (1) will give a better second approximation than Newton's method, subject to our assumptions on $f(x) - c$ and $\log f(x) - \log c$. This is easily seen from a graph.

Mr. Kennedy's numerical example, $x^x = 32.46$ is of this type. His first approximation, 3, with formula one, gives a second approximation 3.087+, which is too large, though it becomes a little too small when only the first three decimal figures are retained. From the statements of the last paragraph we know that Newton's method, applied to $x^x = 32.46$, would give a larger and poorer second approximation. Further, formula (1), simplified to $(a + \log c)/(1 + \log a)$, is easier to use than Newton's method.

Another example satisfying the same hypotheses is $e^x + x = 18.56$, where 3 is a good (too large) first approximation to the real root. Newton's method gives 2.785 for the next approximation. Formula (1), with only a little more labor, gives 2.761, which is only .001 too large. Of course, the general theory does not tell this, only that formula (1) will give a better approximation. We may point out, however, that the formula gives the exact value of the root as the second approximation when the equation is $e^{kx} = c$, and can be expected to be especially satisfactory when the equation is similar (graphically) to this.

If the constant in the last example is changed to a smaller number, say 7.71, the situation changes somewhat. The graph of the function $\log (e^x + x) - \log c$ is now concave downward near the real root, and if we take 2 as our first approximation, formula (1) give a second approximation a little too small. However, by the remark of the last paragraph, we will expect the approximation to be close. It is, in fact, 1.7795, wereas 1.7800 is correct to 4 decimal places. Newton's method, applied to $e^x + x = 7.71$, gives 1.8 as second approximation.

It may also be interesting to consider a few polynomial equations. The equation $x^3 - 6x^2 + 12x - 12 = 0$ satisfies all of our original hypotheses near its real root. If 4 is taken as a first approximation to the real root, Newton's method gives 3.67 as the next approximation, while formula (1) gives 3.616. The real root is actually $2 + 4^{1/3}$, or about 3.58740.

The equation $x^3+2x-20=0$ illustrates a little different situation. The graph of the function $x^3+2x-20$ is concave upward near the real root, while that of $\log(x^3+2x)-\log 20$ is concave downward.[1] The real root appears to be closer to 2 than 3, but using 2 as a first approximation, and Newton's method, we will have our second approximation too large. It is, in fact, 2.57. The poorer first approximation 3 gives a better value, 2.55. Formula (1) has this advantage; we can start with the better first approximation, 2, and be sure in this case that our second approximation has *not* passed beyond the root. We get in fact 2.44, the correction being 6/7 of the difference $\log 20-\log 12$, or $\log 5-\log 3$. The root is 2.46955, to 5 places.

This last type of equation arises many times. We can find a close approximation easily, but the next approximation, by Newton's method, lies on the other side of the root, and perhaps a considerable distance. In some cases formula (1) will keep us on the same side of the root. Again, as in Mr. Kennedy's example, it may take us beyond the root, but not so far as Newton's method.

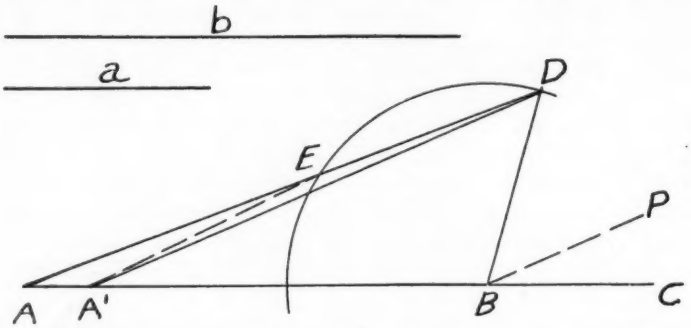
If we apply formula (1) to the equation $x^n=c$, we have the following interesting and perhaps useful

THEOREM. If a is smaller than the real nth root of c, then $a[1+(\log c)/n-\log a]$ is a better approximation to the root than a. Taking $c=10$, $n=3$, $a=2$, the theorem gives 2.149. If we call this 2.15, realizing it is too small, and take it as a, the formula gives 2.15443, which is correct to five decimal places.

ANGLE DIVISION

By E. C. KENNEDY, Univ. of Texas, College of Mines

Any given angle may be divided approximately by ruler and compass into any given ratio and to any desired degree of accuracy by the method described below. By way of illustration let us consider a given angle CBD and two given



[1] To find the second derivative of $\log f(x)$ conveniently, factor $f(x)$ if possible. If $f(x)$ is a polynomial with real, linear factors, the second derivative of $\log f(x)$ is always negative. This general theorem handles many cases, though we have purposely avoided it in our illustrations.

magnitudes a and b. Let it be required to find a point P so that $\angle CBP/\angle PBD = a/b$.

Method: Make $BD=a$, $BA=b$, $EA'=b-a$. Draw BP parallel to $A'D$. Then $\angle CBP/\angle PBD=a/b$, approximately. The accompanying figure is self-explanatory. For greater accuracy locate a point A'' from A' just as A' was located from A. That is, let $A'D$ cut circle at E'. Make $E'A''=EA'=(b-a)$, A'' being on AB. This may be repeated any number of times—the error rapidly approaching zero.

To trisect CBD make $BA=2BD$ and proceed as above. To obtain the angle $CBD/(2+\sqrt{2})$, let $BD=1$ and $BA=1+\sqrt{2}$, etc.

The following data give some idea concerning the accuracy of this method in the case of trisection. e, e', e'' represent the error.

CBD	A	A'	A''
$\pi/12$ rad.	$e=-0.0'\ 45.96''$	$e'=+0.47''$	$e''=-.0025''$
1 rad.	$e=-46'\ \ 16.40''$	$e'=+8'3.50''$	$e''=-1'19.87''$

An approximate measure of e' is given by

$$e' = 4\theta^5/3^7$$

which is quite accurate for small values of θ. Both e' and $\theta(=CBD)$ are measured in radians. This excellent approximation was suggested to the writer by Prof. Dunkel.

Thus it seems that any angle of about 60°, say, may be trisected with great accuracy by either (1) bisecting the given angle two or three times (thus dealing with a small angle) and locating A' or (2) by locating A''' or A^{IV} as described above.

The writer suggests that someone find a measure of $e^{(n)}$ for $CBD=15°$.

Additional Marginal Notes

By L. S. Johnston, University of Detroit

The title of this note is frankly and appropriately suggested by Professor Hildebrandt's "*Marginal Notes*" published in this Monthly, April, 1929. Like the material of that note, this also is culled from classroom experience. No claims for priority are made except where specifically so stated.

I. *Analytic Geometry.*

1. Several classic theorems of Projective Geometry which are either not provable at all by high school geometry or if provable are very difficult can be easily exhibited, and, by the more ambitious and able student, even actually

demonstrated, by use of the ordinary theory of the straight line and the circle in Analytic Geometry. Illustrative examples of such theorems are not at all difficult to set up, and I have found that by such exercises many students are attracted to the subject, and to mathematics as a whole, to a degree rarely found when only the more conventional exercises are used.

Some theorems which can be so approached are: Existence of the Euler Line for any triangle- that is, the collinearity of the centroid, the circumcenter, and the orthocenter; existence of the Nine Point Circle; Pascal's Hexagon Theorem and its dual, the Brianchon Theorem, using the degenerate two line conic and the circle rather than the general conic; Desargues' Theorem and the converse; Pascal's Pentagon Theorem and its use in drawing the tangent to any circle (or any conic, for that matter) at any point on it without using the center of the circle or any perpendicular lines.

2. *Constructions on the Ellipse and Hyperbola.* The following theorems on the ellipse are very easily proved, and suggest constructions useful in drawing the curves.

i) Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$, $c^2=a^2-b^2$, foci at F and F', X intercepts at A and A' as usual. The points $(0, a)$, $(c, b^2/a)$, and $(a, a-c)$ are collinear on the tangent to the ellipse at $(c, b^2/a)$.

ii) The points $(-a, 0)$, $(0, a-c)$, and $(c, b^2/a)$ are collinear, and, by (i) above, the line through $(c, b^2/a)$ and $(a, a-c)$ is tangent to the ellipse at $(c, b^2/a)$.

iii) If R and R' are respectively on the lines $x=a$ and $x=-a$, and FR is perpendicular to FR', the line RR' is tangent to the ellipse, as is well known. If FS be drawn so that the angles RFS and RFA are equal, the intersection of RR' and FS is the point of contact of RR' with the ellipse. This is proved in the writer's paper "*On Certain Sequences of Conics and Associated Sequences of Numbers*," this MONTHLY, April, 1929.

These three theorems are easily modified to apply to the hyperbola—the only modification, indeed, being that in the first theorem the point $(0, -a)$ is used instead of $(0, a)$.

I am not aware that any of these theorems have been mentioned in any elementary text.

3. *Simplification of Conics by Rotating Axes.* There is an increasing tendency to omit this topic from elementary courses, probably due to the difficulties inherent in the trigonometric and numerical calculations involved. Most of these difficulties can be eliminated if certain relations given in most elementary texts be used.

Let the translated form of the conic be $Ax^2+Bxy+Cy^2=F$, with the form $A'x^2+C'y^2=F'$ to be derived. The relations

$$A' + C' = A + C, \tag{1}$$

$$A' - C' = [(A - C)^2 + B^2]^{1/2}, \tag{2}$$

the sign of the square root to be taken to agree with the sign of B, and

$$\tan 2\theta = \frac{B}{A - C} \tag{3}$$

are well known, and are set as exercises in most texts. Tan θ is usually calculated from the identity $\tan^2\theta = (1-\cos 2\theta)/(1+\cos 2\theta)$.

The entire simplification can be performed without using θ at all, by using relations (1) and (2) above. Tan θ is most easily calculated from the identity $\tan\theta = (\tan 2\theta)/(1+\sec 2\theta)$, which, compared with (3) above, gives

$$\tan\theta = \frac{B}{A - C + A' - C'}. \tag{4}$$

II. *Calculus.*

1. Let $\rho = f(\theta)$ be the equation of a plane curve in polar coordinates; let ψ be the angle between the radius vector to any point (ρ_1, θ_1) and the tangent to the curve at this point, measured in positive direction from the radius vector; let τ be the angle between the tangent line to the curve and the polar axis, measured in positive direction from the polar axis. The usual method of deriving the formula for ψ in terms of ρ and θ seems unnecessarily laborious. I have found the following approach more easily handled in the classroom: From the relations $\rho^2 = x^2 + y^2$, $\tan\theta = y/x$, $\tan\tau = dy/dx$, we have

$$\tan\psi = \frac{\tan\tau - \tan\theta}{1 + \tan\tau\tan\theta} = \frac{xdy - ydx}{xdx + ydy}$$

$$= \sqrt{x^2 + y^2}\left(\frac{\dfrac{xdy - ydx}{x^2}}{1 + \dfrac{y^2}{x^2}}\right) \Big/ \left(\frac{xdy + ydx}{\sqrt{x^2 + y^2}}\right) = \frac{\rho d\left(\text{arc tan}\dfrac{y}{x}\right)}{d\rho}$$

$$= \rho\frac{d\theta}{d\rho}.$$

2. Let the parametric equations of motion in a plane be $x = x(t)$, $y = y(t)$, and let v_x, v_y, $v(=v_t)$, α_x, α_y, α, have their usual meanings; let

α_t = tangential acceleration
α_n = normal acceleration
v_R = radial velocity (velocity component along the radius vector)
α_R = radial acceleration
v_T = transverse velocity (velocity component normal to radius vector)
α_T = transverse acceleration.

The following simple relations are easily derived and applied:

$$\alpha_t = \frac{v_x\alpha_x + v_y\alpha_y}{v}; \quad \alpha_n = \frac{\begin{vmatrix} v_x & v_y \\ \alpha_x & \alpha_y \end{vmatrix}}{v}$$

$$v_R = \frac{xv_x + yv_y}{\sqrt{x^2+y^2}}; \quad v_T = \frac{\begin{vmatrix} x & y \\ v_x & v_y \end{vmatrix}}{\sqrt{x^2+y^2}}$$

$$\alpha_R = \frac{x\alpha_x + y\alpha_y}{\sqrt{x^2+y^2}}; \quad \alpha_T = \frac{\begin{vmatrix} x & y \\ \alpha_x & \alpha_y \end{vmatrix}}{\sqrt{x^2+y^2}}.$$

III. *Differential Equations.*

Consider the equation

$$\sum_{j=0}^{n} a_j \frac{d^j y}{dx^j} = f(x), \; d^0y/dx^0 \equiv y.$$

One method of calculating the particular integral which is frequently used is to differentiate the given equations enough additional times (say k times) to make it possible to find constants $b_0, b_1, \cdots b_k$ (if such constants exist, of course) such that $\sum_{j=0}^{k} b_j f^j(x) = 0$, and to use the resulting auxiliary equation $\sum_{j=0}^{n+k} A_j r^j = 0$. It is easily shown that this last equation can be factored into $(\sum_{j=0}^{k} b_j r^j)(\sum_{j=0}^{n} a_j r^j) = 0$, and hence that the particular integral can be found by using the equation $\sum_{j=0}^{k} b_j r^j = 0$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Differential Equations. By Forest Ray Moulton. New York, The Macmillan Company, 1930. xv+395 pages. $5.50.

This book is of very general interest and is at the same time a personal revelation of its author. It portrays Professor Moulton as one whose chief interests are practical, who regards differential equations primarily as tools in the solution of problems in astronomy and physics, but also as one who scorns the use of doubtful or unworthy tools and finds a keen satisfaction in the beautiful perfection of rigorous mathematical demonstration. It is this rare combination of application and theory, which sets this book apart.

In scope, the book differs from that of any other known to the reviewer.

Like Ince's book, it is limited to the study of ordinary differential equations. But unlike Ince it does not consider such problems as the characterization of differential equations by their singular points nor differential systems containing general boundary conditions as distinguished from initial conditions. Nor does it review the solution of differential equations by elementary methods. With these limitations it is quite comprehensive and presents most of the generally useful methods of solution. Numerous references to the eight appendices on topics in the Theory of Functions enable a good student to read the book without much more background in analysis than a year of advanced calculus.

Each existence proof of course does double duty. It is a proof and a careful one, but it is also a method or formula for the actual solution of a definite problem. To the author, perhaps, neither function seems more important than the other. It is doubtless to prevent the *proof* from overshadowing the *method* that theorems are rarely set apart and stated formally as theorems. Though this style does allow the theory and the formulae of solutions to develop coordinately, it increases the difficulty of reading the book and makes it less convenient for reference.

Chapter I, Generalities and Definitions, defines a normal system of differential equations:

$$\frac{dx_1}{dt} = f_1(x_1, \cdots, x_n; t),$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{dx_n}{dt} = f_n(x_1, \cdots, x_n; t).$$

Such systems furnish the material for treatment in the book. Solutions and integrals of normal systems are defined and illustrated. This chapter, like the others, contains a list of suggestive problems.

Chapter II, Analytic Differential Equations, assumes the functions f_i to be analytic in all the variables, and adjoins the initial conditions $x_i(t_0) = a_i$. It sets up the formal solution in power series, and proves the convergence by the method of dominant functions.

Chapter III, Differential Equations Analytic in a Parameter, first sets up a solution as a power series in μ for the case of equations in the form $dx_i/dt = \mu f_i(x_j : \mu : t)$ and then considers more general systems.

Chapter IV, Variation of Parameters, is a general discussion of this method of transformation with special application to linear equations.

Chapter V, Integrals of Differential Equations, deals with the existence under suitable hypotheses of a fundamental set of integrals and with the use of known integrals in reducing the order of an equation.

Chapter VI, Analytic Implicit Functions, treats of the existence and formulation of solutions of systems of the form

$$a_{11}x_1 + \cdots + a_{1n}x_n = c_1 y + \phi_1(x_1, \cdots, x_n; y)$$

. . . . . .

$$a_{n1}x_1 + \cdots + a_{nn}x_n = c_n y + \phi_n(x_1, \cdots, x_n; y)$$

(where the ϕ_i are power series in the x_j and y and contain no terms of lower degree than the second) under various hypotheses on the determinant $|a_{ij}|$.

In Chapter VII, The Problem of Elliptic Motion, we find the first of the major applications which give the book its real character. The problem of two bodies is set up in the form of a differential system. The nature of the orbit and the period are first determined directly. Then the methods of Chapters II and III are used to form the complete solution as a power series in the variable of time, and also in terms of the eccentricity.

Chapter VIII, The Sine-amplitude Function, develops the characteristic properties of the sine-amplitude function partly from the corresponding elliptic integral but mainly as properties of the solution of the equivalent differential equation.

Chapter IX, The Deviations of Falling Bodies, gives the author's own discussion of the deviations of a falling body from a plumb line.

Chapter X, The Damped Gyroscope, emphasizes the necessity of choosing the method of solution to suit the problem. The method of variation of parameters is exemplified in studying the nutations of the axis.

In Chapter XI, The Method of Successive Integrations, the process of successive approximations is established in a general way by means of a variation in the form of the first integral.

Chapter XII, Numerical Solution of Differential Equations, is an exceedingly interesting chapter which expounds the author's own method of numerical solution previously developed in his work on Ballistics. The method, in brief, consists in approximating to the solution in a very small interval by the method of successive approximations, and then extrapolating to get initial values on an adjoining interval. For solutions of this type, the author's arrangement of the work is of prime importance.

In Chapter XIII, The Cauchy-Lipschitz Process, this classical method is explained and compared with the method of Chapter XI.

Chapter XIV, Generalities on Linear Differential Equations, defines a fundamental set of solutions. It also defines the equations for variations of the solution corresponding to variations in initial conditions.

Chapter XV, Linear Differential Equations with Constant Coefficients, is unusual in adopting the method of Nyswander rather than the customary treatment. The chief advantage is in simplifying the routine of actual calculation. The characteristic determinant is reduced in a novel manner to a normal form that is always applicable.

Chapter XVI, Applications of Linear Differential Equations, gives applications of the problem of stability of planetary orbits, and to an electric circuit with a periodic imposed electromotive force.

Chapter XVII, Linear Differential Equations with Periodic Coefficients, extends the methods of Chapter XV to the equations described in the title. Identities are introduced analogous to those of Nyswander used in Chapter XV, and the characteristic equation is reduced in a manner similar to the one employed there.

In Chapter XVIII, Differential Equations of Infinitely Many Variables, necessary restrictions are placed on the functions of the system, a formal solution is set up, and convergence established. Interesting applications are then made to an infinite universe extending in the one sense to subelectrons and on the other to super-galaxies of all orders. On the basis of postulates laid down it is established that the individual stars have determinable motions with respect to arbitrary axes. The system occupies infinite space in such a way that the sums of components of attraction on each star are finite.

In the main the details of the book have been well handled. It is not entirely free from lapses in notation and expression. For example, on page 98 the same letter is used in a single equation to represent the eccentricity of an ellipse and the Napierian base of logarithms. On page 117 the equations $t=\frac{1}{2}\sqrt{-1}\,P_2$ and $t=\frac{1}{2}P_1$ do not properly describe the lines in question. On page 249 the assumption should be $\delta=0$ instead of $\delta\neq 0$.

The book is a valuable and important contribution and is recommended to students of mathematics, physics, and astronomy. It will be relished by all who have an appetite for rigorous mathematical deduction flavored with the wholesome tang that comes from keeping close to a basis of physical application and numerical computation.

RAYMOND W. BRINK

Elèments de Trigonomètrie Sphérique. By G. Papelier, Paris, Librairie Vuibert, 1930. 162 pages. Paper, 20 francs.

This small inexpensive book contains a surprising amount of material which is presented in a most excellent manner. The first five chapters—which is less than half of the book—are devoted to general remarks on spherical triangles, relations among the elements of a spherical triangle, properties of spherical triangles, solutions of right spherical triangles, and solutions of triangles in general. All cases are carefully and fully discussed. A numerical example is generally given for the sake of fuller illustration. The material covered here is more extensive than that contained in the ordinary plane and spherical trigonometry text book.

The last five chapters are devoted to a discussion of the escribed, inscribed, and circumscribed circles of spherical triangles; medians, bisectors, and altitudes of spherical triangles; formulas relative to spherical excess; several diverse applications; and the solution of triangles which come outside the standard cases. These chapters all contain material that is of much general interest and would prove stimulating to any college student who is interested in further read-

ing in spherical trigonometry. Especially is this true of the material leading up to the theorems of Menelaus and Ceva.

The book is practically void of exercises, there being only eighteen, which are given at the end of the book along with a résumé of the most important formulae that have been developed.

H. M. HOSFORD

A LETTER TO THE EDITOR

DEAR SIR:

In the recent comment in your columns on my *Table Slide Rule* it was stated that this Rule requires a large space for efficient use. Permit me to point out that the dimensions of the plates are $8\frac{1}{2}'' \times 11''$, so that they will fit into standard size loose-leaf notebooks. As published the plates are not punched because there is no uniformity in the punching of notebook paper. They may however be punched locally so that they may fit the notebooks used.

When carried in a notebook the slide rule is immediately available and is found to be very convenient.

Sincerely yours,
J. P. BALLANTINE

June 2, 1932.

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

A.

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research in the preparation of papers in the field of mathematical science to be presented at its regular meetings.

CHAPTER REPORTS

1931–1932

Pi Mu Epsilon of the University of California at Berkeley

The officers for the academic year 1931–1932 were elected by the members at their last regular meeting, on April 21, 1931. They were: Mr. W. L. Hutchings, Director; Miss C. M. Daly, Vice Director; Miss D. A. Wilson, Secretary; Mr. A. L. Buckman, Treasurer; Mr. R. M. Robinson, Librarian.

There are fifty-six active members, eighteen of which were initiated October 3, 1931, and eight on March 19, 1932.

Monthly meetings were held in Wheeler Hall and the following papers were presented.

August 26, 1931: "Elementary hyperbolic functions considered from a geometric viewpoint" by Mr. W. L. Hutchings.

September 17, 1931: "Pastures of wonder" by Mr. A. H. Diamond.

October 14, 1931: "Special solution of the three body problem" by Mr. A. L. Buckman.

November 4, 1931: "Seismology" by Mr. E. C. Robison.

December 5, 1931: "Regular geometrical figures in four-dimensional space" by Mr. R. M. Robinson.

January 27, 1932: "Auto-polar curves" by Dr. D. C. Duncan.

February 24, 1932: "A general space transformation" by Mr. E. R. Mathews.

March 29, 1932: "Short topics" by Mr. V. H. Ivanoff.

April 20, 1932: "Self-projective curves in two-dimensional space" by Dr. K. Woo; Election of officers for 1932–1933.

The annual picnic was held at John Hinkel park on Friday afternoon, May 6, 1932.

ELLA S. ROSENBERG, *Secretary*

Pi Mu Epsilon of Bucknell University

The officers for 1931–1932 were: Professor J. S. Gold, Director; Anthony Chernefski, Vice Director; Francis Knights, Secretary; Hulda Dimm, Treasurer. The officers are elected by a majority vote near the end of the Spring semester.

The meetings and programs were as follows:

October 5, 1931: "Castigliano's theorem and the theorem of least work" by Professor D. M. Griffith.

November 2, 1931: Initiation of new members and banquet.

December 7, 1931: "Some 'new methods' (?) of trisecting an angle" by Professor C. H. Richardson.

January 11, 1932: "The nine-point circle" by Miss Lulu Miller.

February 8, 1932: "The approximate solution of equations" by Miss Frances Knights.

March 4, 1932: "The work of an actuary" by Mr. James Herman, Assistant Actuary of the Metropolitan Life Insurance Company.

April 4, 1932: "Heaviside's operational calculus" by Professor George Irland.

May 13, 1932: Annual picnic and election of officers.

FRANCES KNIGHTS, *Secretary*

The Mathematics Club of Creighton University

This club was organized in 1929 by Professor Alvin K. Bettinger for the purpose of bringing together students interested in mathematics and giving them an opportunity for discussing current topics in mathematics. Any university student interested in mathematics is eligible and a member in the organization, the affairs of which are conducted by an executive committee which during the following year was composed of Paul Schneider, President; Hugh M. Schwaab, Vice President; Joseph Solomonow, Secretary-Treasurer; Alvin K. Bettinger, Faculty Moderator. There are 27 active members at present.

The meetings and programs were as follows:

September 30, 1931: "Descartes" by Walter Smith; "Mathematical wrinkles" by Paul Schneider; "Desargue's theorem" by Mr. Frank E. Marrin.

October 28, 1931: "Euclid" by Robert Swanson; "Infinite series" by Joseph C. McCarthy; "Non-differentiable continuous functions" by Professor Alvin K. Bettinger.

November 19, 1931: "Thales and Appolonius" by William Murphy; "Graphical representations"

by John Dittrick; "Practical applications of mathematics to paths and projectiles" by Mr. Wendell A. Dwyer.

December 16, 1931: "Pascal's theorem" by Paul Schneider; "Brianchon's theorem" by Hugh M. Schwaab.

January 20, 1932: "Mathematics and music" by Joseph Solomonow; "Telegraph, telephone and mathematics" by Mr. Blakeslee of Northwestern Bell.

February 17, 1932: "Vector equations" by Olga Dyba; "Pythagoras" by Walter Smith; "The fundamental theorem of algebra" by Mr. John Fitzpatrick.

March 18, 1932: "Hyperbolic functions" by Professor Alvin K. Bettinger; "Poles and polars" by John Dittrick.

April 27, 1932: "Transcendentals" by Basil Lazure; "Families of curves and surfaces" by John Meters; "The nine-point circle" by Mr. Frank E. Marrin.

May 19, 1932: "Abel and Galois" by Glenn Rhoades; "Harmonic curves" by Mr. Zimmerman, S. J.; "Principles of astronomy" by Mr. Wendell A. Dwyer. This meeting was followed by a trip through the Creighton observatory with explanations of the apparatus by Mr. Dwyer.

JOSEPH SOLOMONOW, *Secretary*

The School of Crotona of Carleton College

The School of Crotona is a local honorary mathematics society which was organized three years ago for the purpose of stimulating an interest in the mathematical world.

To be eligible for membership, one must have completed at least one year of study in differential and integral calculus, as well as shown general mathematical ability. A comprehensive oral examination is given to qualified applicants by the faculty of the department of mathematics.

Three members, who constitute the Triad, are chosen from the Society each year to form the governing body of the Mathematics Club, a club whose meetings are open to all students interested in that subject. The members of the Triad this year are: William Conley, President; Frederick Schurmeier, Secretary-Treasurer; and Robert McCallum.

Other members of the School of Crotona are Philip Nason, Eleanor Walker, Impi Aho, and Edward Tomastic. Members of the Faculty are Dr. Gingrich and Dr. White.

The meetings and programs of the Mathematics Club were as follows:

October 2, 1931: "Symmetry in mathematics" by Dr. Gingrich of Carleton College, Chairman of the Department of Mathematics and Editor of Popular Astronomy.

November 19, 1931: "Hyperbolic functions" by Dr. Carlson, Professor of Mathematics, St. Olaf's College.

January 14, 1932: "Pascal's triangle" by Mr. William Conley, President of the Mathematics Club.

March 22, 1931: "Mathematics in business" by Dr. Larson of the Faculty of Carleton College.

April 12, 1932: "Mathematics of investment" by Dr. Hart, Professor of Mathematics in the University of Minnesota.

F. A. SCHURMEIER, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND WM. FITCH CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and de-

manding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E1. *Proposed by H. E. Slaught, University of Chicago.*

Find all the unknown digits, represented by x's, in this exact division.

$$
\begin{array}{l}
x\,x\,x\,)\;x\,x\,x\,x\,x\,x\,x\,x\,x\;(x\,7\,x\,x\,x \\
\quad\;\underline{x\,x\,x\,x} \\
\qquad\;\;x\,x\,x \\
\qquad\;\;\underline{x\,x\,x} \\
\qquad\;\;x\,x\,x\,x \\
\qquad\quad\;\underline{x\,x\,x} \\
\qquad\qquad\;x\,x\,x\,x \\
\qquad\qquad\;\underline{x\,x\,x\,x}
\end{array}
$$

E2. *Proposed by R. K. Morley, Worcester Polytechnic Institute.*

In finding the volume cut from the sphere $x^2+y^2+z^2=9$ by the cylinder $x^2+y^2=3x$, one may use cylindrical coordinates and choose as element of volume a column of cross-section dr by $rd\theta$ and of height $z=(9-r^2)^{1/2}$. The volume appears computable either by

$$V_1 = 2\int_{-\pi/2}^{+\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}r\,dr\,d\theta,$$

where symmetry with respect to the xy-plane only is used, or by

$$V_2 = 4\int_0^{\pi/2}\int_0^{3\cos\theta}(9-r^2)^{1/2}r\,dr\,d\theta,$$

where symmetry with respect to the xz-plane is also used. But upon integration it appears that V_1 exceeds V_2 by twenty-four cubic units. Which (if either) is correct, and why?

E3. *Proposed by Wm. R. Ransom, Tufts College.*

Has the locus $y=x^x$ a highest point in the second quadrant?

E4. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

What is the simplest way to cut a wooden block 1 ft. $\times$1 ft. $\times$2 ft. into pieces which may be reassembled into a solid cube?

E5. *Proposed by Wm. Fitch Cheney, Jr., Connecticut Agricultural College.*

How may the total surface of a sphere be divided into the largest possible number of congruent pieces, if each side of each piece is an arc of a great circle less than a quadrant?

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3566. *Proposed by V. Ivanoff, San Francisco, Calif.*

A perpendicular CM_0 is drawn to the line segment M_0M_n and the points $M_1, M_2, \cdots, M_{n-1}$ are taken on M_0M_n so that $\angle M_0CM_1 = \angle M_1CM_2 = \cdots = \angle M_{n-1}CM_n = \alpha$. By parallel displacement of the vectors $CM_i (i = 2, 3, \cdots, n)$ a broken line $CM_1M_2'M_3' \cdots M_n'$ is formed with $M_1M_2' = CM_2$ and $M_{i-1}'M_i' = CM_i (i = 3, 4, \cdots, n)$. Find the locus of the vertices of this broken line if $CM_0 \to 0$ and $\alpha \to 0$ in such a way that $CM_0/\alpha = K$, K constant.

3567. *Proposed by Leverett Davis, University of Washington.*

Find the line through a given point that shall form with two given lines a triangle of minimum area. Characterize the solution geometrically.

3568. *Proposed by Orrin Frink, Pennsylvania State College.*

What is the convex region of greatest area which contains a given triangle T but contains no triangle of greater area than T?.

3569. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given line s to draw a plane cutting the faces of a given dihedral angle along the lines a, b, and the bisecting planes of this dihedral angle along the lines c, d, so that the lines c, d shall be the bisectors of the angles formed by a, b. The problem has, in general, two solutions. What are the special cases?

3570. *Proposed by Norman Anning, University of Michigan.*

Given five points in the same three-space but otherwise independent, construct the point in which the join of two of them intersects the plane of the other three.

3571. *Proposed by Lester R. Ford, Rice Institute.*

What should one pay in order to receive $1, 8, 27, \cdots, n^3, \cdots$, dollars at the end of $1, 2, 3, \cdots, n, \cdots$, years, the interest rate being six percent?

SOLUTIONS

3365. [1929, 168]. *Proposed by R. E. Gaines, University of Richmond.*

An open pan with a square bottom (a frustum of a pyramid) having a given

total surface is to be made so as to have a maximum content. Find its dimensions.

Solution by Otto Dunkel, Washington University.

The pans considered in this solution have the form of a frustum of a regular pyramid with a square base. Let x be the length of the side of the square base of the pan; y, that of the open top; z, the slant height of the lateral sides; h, the altitude of the pan; a^2, the superficial area; and V, its content. We have the following relations;*

$$(1)\qquad a^2 = x^2 + 2z(x + y);\ 4h^2(x + y)^2 = (a^2 - y^2)(a^2 + y^2 - 2x^2);$$
$$6V = [(a^2 - y^2)(a^2 + y^2 - 2x^2)]^{1/2}\frac{x^2 + xy + y^2}{x + y}.$$

If a pan V_1 has $y<x$, then a pan V_2 of the same height whose base has the dimensions of the top of V_1 and whose top has the dimensions of the base of V_1 has the same content as V_1, but it has less superficial area. Hence there is a pan V_3 with a form similar to that of V_2 and with the area of V_1 whose content is greater than that of V_1. This shows that we need to consider only the cases $y \geqq x$. Since in such cases y^2 is the projection of a^2 on the base, we must have also $y \leqq a$. Hence we have a region defined by $a \geqq y \geqq x \geqq 0$; this region is an isosceles right triangle. Along the hypotenuse we have boxes, i.e., pans with vertical sides; along the leg for which $x=0$ we have the cases where the base reduces to a point, or funnel forms; while along the leg $y=a$ the height is zero and $V=0$. The maximum volumes in the first two cases will be denoted by V_b and V_f, respectively. We easily find

$$(2)\qquad V_b = 2^{-1}3^{-3/2}a^3, \quad x = y = 3^{-1/2}a;$$
$$V_f = 2^{-1/2}3^{-7/4}a^3, \quad x = 0,\ y = 3^{-1/4}a;$$
$$V_b < V_f.$$

If we define $V=0$ for $x=y=0$, then V is a continuous function of x and y in the closed triangular region. It will appear later that V has values within the region greater than its maximum on the boundary; and hence V has an absolute maximum at some point, or points, within the region. At such points the partial derivatives V_x and V_y, which exist at all points within the region, must be zero. It will be shown that there is only one point within the region where the derivatives vanish simultaneously. Therefore this point must give an absolute maximum. From these considerations we see that we may reject all factors which vanish only on the boundary or outside the region in seeking the solution of $V_x = V_y = 0$ at an interior point. For convenience set

$$(3)\qquad R = (a^2 - y^2)(a^2 + y^2 - 2x^2);$$
$$R_x = -4x(a^2 - y^2);\ R_y = -4y(y^2 - x^2).$$

The equations $V_x = V_y = 0$ give

$$R(x+2y) - 2(a^2-y^2)(x+y)(x^2+xy+y^2) = 0, \tag{4}$$
$$R(y+2x) - 2(y^2-x^2)(x+y)(x^2+xy+y^2) = 0.$$

It follows at once from these two equations that

$$(y+2x)(a^2-y^2) = (x+2y)(y^2-x^2). \tag{5}$$

The first equation of (4) gives at once

$$(x+2y)(a^2+y^2-2x^2) = 2(x+y)(x^2+xy+y^2); \tag{6}$$

and adding (6) and (5) we have after slight reduction

$$a^2 = x^2+xy+y^2. \tag{7}$$

Eliminating a^2 from (5) by (7) we find

$$2y^2 - 2xy - 3x^2 = 0. \tag{8}$$

Set $y=rx$ in (8), then

$$2r^2 = 2r+3,\ 2r = 1+7^{1/2},\ y = rx. \tag{9}$$

The first equation in (9) is useful for reducing higher powers of r to linear expressions in r before inserting the numerical value of r. From (7) and (9) there results

$$x^2 = \frac{a^2}{1+r+r^2} = \frac{2a^2}{4r+5} = \frac{2a^2(7-2\cdot 7^{1/2})}{21}, \tag{10}$$
$$y^2 = \frac{a^2(14-7^{1/2})}{21}.$$

The first equation in (1) with (7) gives

$$z = \tfrac{1}{2}y. \tag{11}$$

In order to obtain h it is simpler to return to the right triangle for h:

$$h^2 = \left(\frac{y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2 = \tfrac{1}{4}x(2y-x) = \tfrac{1}{4}(2r-1)x^2 = \tfrac{1}{4}7^{1/2}x^2, \tag{12}$$
$$= \frac{a^2(7^{1/2}-2)}{6}.$$

We now find from (7) and (12) the maximum

$$V_p = \frac{a^3}{3}\left(\frac{7^{1/2}-2}{6}\right)^{1/2}. \tag{13}$$

It is easily shown that $V_p > V_f$. The numerical results are as follows:

$$x = .403378a, \qquad y = .735308a, \qquad z = .367654a,$$
$$V_p = .1093543a^3, \quad V_f = .103401a^3, \quad V_b = .096225a^3.$$

The sides of the pan make with the base an angle $\theta = 2 \text{ arc cot } 7^{1/4} = 63° 9' 55''$.

3469 [1931, 50]. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

A Note by the Editors. A solution of this problem by R. Goormaghtigh has been printed in this Monthly [1932, 53–54], and the solver has recently sent in the generalization in the note below.

II. *Solution by R. Goormaghtigh, Bruges, Belgium.*

In our article "*Solution géométrique du problème fondamental de la Géométrie intrinsèque plane*" (*Mathesis*, 1929, p. 53) we have given a geometrical solution of the following general problem:

If a moving point Q has the coordinates x, y with respect to the tangent and the normal at a moving point P on a plane curve (P), x and y being functions of the inclination angle ϕ of the tangent to (P) at P with respect to a fixed axis, give a geometrical construction for the normal and the center of curvature at Q to the locus (Q) of Q.

Let x', y' and x'', y'' be the two first derivatives of x, y with respect to ϕ; let further C be the center of curvature of (P) at P and C' the center of curvature of the evolute of (P) at C.

The normal to (Q) at Q passes through the point T having x', y' as coordinates with respect to PC, CC'.

If S is the point having x'', y'' as coordinates with respect to the tangent and the normal to the curve (C'), second evolute of (P), the center of curvature q of (Q) at Q lies on the straight line joining the mid-point of TS to the point where QS meets the perpendicular erected on QT at T.

The point q can also be obtained as follows:

The point q is the mid-point of the distance between T and the harmonically conjugate of T with respect to Q and the projection of S on QT.

A Note by Otto Dunkel. Since *Mathesis* may not be accessible to many readers, as is the case with the writer, a proof of this theorem will be given, and vector analysis will be used to show the advantage of this form of analysis for curvature problems. Let $\boldsymbol{t}$ and $\boldsymbol{n}$ be the unit vector tangent and normal for (P) at P, where the positive direction of $\boldsymbol{t}$ is that of increasing arc length s and that of $\boldsymbol{n}$ is along $PC = \rho$; let $CC' = \rho_1$. The unit tangent and normal to (C) at C are $\boldsymbol{n}$ and $-\boldsymbol{t}$; for (C') at C', they are $-\boldsymbol{t}$ and $-\boldsymbol{n}$. It will be seen from a figure that

$$QT = (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t}; \tag{1}$$

$$ST = (\rho_1 - y' + x'')\boldsymbol{t} + (x' + y'')\boldsymbol{n}. \tag{2}$$

Let $\boldsymbol{r}$ and $\boldsymbol{r}_2$ be the vectors of P and Q drawn from a given origin, so that

$$\boldsymbol{r}_2 = \boldsymbol{r} + x\boldsymbol{t} + y\boldsymbol{n}. \tag{3}$$

After differentiation with respect to s, (3) gives

$$s_2' \boldsymbol{t}_2 = \boldsymbol{t} + \kappa(x'\boldsymbol{t} + y'\boldsymbol{n}) + \kappa(x\boldsymbol{n} - y\boldsymbol{t});$$

where the accents refer to differentiation with respect to ϕ only in the cases of x and y, while with other quantities they denote differentiation with respect to s; $\kappa = \rho^{-1} = d\phi/ds$; $s_2' = ds_2/ds$ is taken as positive; and where the unit vector tangent $\boldsymbol{t}_2$ has the direction of increasing arc s_2. The equation above may be written

$$s_2' \rho \boldsymbol{t}_2 = (\rho + x' - y)\boldsymbol{t} + (x + y')\boldsymbol{n}. \tag{4}$$

Since the scalar product $QT \cdot \boldsymbol{t}_2$ is easily seen from (1) and (4) to be zero, it follows that the vector QT is along the unit normal $\boldsymbol{n}_2$ to (Q) at Q. Let the absolute value of the right side of (1) and (4) be l. Then we may write

$$\begin{aligned} l\boldsymbol{t}_2 &= (\rho + x' - y)\boldsymbol{t} + (x + y')\boldsymbol{n}; \; s_2'\rho = l; \\ l\boldsymbol{n}_2 &= (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t}. \end{aligned} \tag{5}$$

In order to find the curvature $\kappa_2 = \rho_2^{-1}$ of (Q) at Q, we differentiate the first equation of (5) with respect to s. There results after slight changes

$$\begin{aligned} \rho l'\boldsymbol{t}_2 + \rho l \kappa_2 s_2' \boldsymbol{n}_2 &= (\rho + x' - y)\boldsymbol{n} - (x + y')\boldsymbol{t} \\ &\quad + (\rho\rho' + x'' - y')\boldsymbol{t} + (x' + y'')\boldsymbol{n}; \\ &= l\boldsymbol{n}_2 + ST; \end{aligned} \tag{6}$$

where $\rho_1 = \rho\rho'$. Taking the scalar product of $\boldsymbol{n}_2$ and the second form of (6) we have

$$l^2\kappa_2 = l + ST \cdot \boldsymbol{n}_2$$

Let the projection of S on QT be M, then the last equation becomes $l^2 = \rho_2(l + MT)$, or

$$\frac{\rho_2 - l}{\rho_2} = \frac{TM}{l}, \text{ or } \frac{Tq}{Qq} = \frac{TM}{QT}, \tag{7}$$

where q is the center of curvature of (Q) at Q.

In order to derive the constructions for q, draw TH perpendicular to QT at T cutting QS in H, and produce TH to K so that $HK = TH$. Draw KS cutting QT in T'. It is obvious that TS, $T'H$ and MK meet in a point, and that Q, T, M, T' form an harmonic set of points. Draw Hq parallel to KS cutting QT' in q. Obviously Hq bisects ST, and q bisects TT'. It will be shown that the q thus

obtained is the desired center of curvature. From the figure we have

$$\frac{Tq}{Qq} = \frac{qT'}{Qq} = \frac{HS}{QH} = \frac{TM}{QT},$$

and hence this q is the same as in (7).

We obtain additional equations by taking the scalar product of t_2 and (6). We have $\rho l' = ST \cdot t_2 = NQ$, where N is the projection of S on the tangent at Q to (Q). This equation with $\rho s_2' = l$ gives

$$\frac{dl}{ds_2} = \frac{NQ}{QM}.$$

3482 [1931, 170]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If the circumscribed and the inscribed spheres of a tetrahedron are concentric, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Solution by Wallace Smith, New River State College, Montgomery, W. Va.

Sphere S is circumscribed about and sphere s is inscribed in the tetrahedron $ABCD$, each having the center O. Since planes ABD, BDC, ADC, and ABC are tangent to sphere s, they are at equal distances from the center of sphere S. Planes equidistant from the center of a sphere intersect the sphere in equal circles. Therefore, triangles ABD, ABC, ADC, and BCD are inscribed in equal circles. The chords AD, AB, and BD are each common to two equal circles. It follows at once that $\angle ACD = \angle ABD$, $\angle BCA = \angle BDA$, and $\angle BCD = \angle BAD$. Therefore, $\angle BCD + \angle BCA + \angle ACD$ equals $\angle BAD + \angle BDA + \angle ABD$ equals two right angles.

A similar proof applies to the other three vertices.

Therefore, the sum of the face angles of each trihedral angle of the tetrahedron is equal to two right angles.

Also solved by V. F. Ivanoff, A. Pelletier, J. Rosenbaum, Otto J. Ramler, and F. Underwood.

A Note by Otto Dunkel. In the above proof the fact that pairs of angles such as ACD and ABD are equal rather than supplementary follows from the fact that each of such face angles is acute. For, the center O of the inscribed sphere s is the intersection of the three planes bisecting the dihedral angles at AB, BC, CA. Thus, if the point of tangency of s with the face ABC is T_d, this point must lie within each angle of the triangle ABC, and, consequently, it must lie within the area of that triangle. Moreover, since O is also the center of S, T_d is the center of the circle circumscribing ABC and lying upon S. Since the center of the circumcircle lies within ABC, all its angles must be acute; and, similarly

all the face angles of the tetrahedron must be acute. It was proved above that the four circumcircles are equal.

Let the other three faces be rotated into the plane of ABC so that the vertex D falls at D', D'', D''', where A and D', B and D'', C and D''', lie, respectively, on opposite sides of BC, CA, AB. As shown above the interior angles at D', D'', D''' are equal, respectively, to the angles A, B, C of triangle ABC. Also $\angle D''AC = C'$, $\angle D'''AB = B'$, where $A + B' + C' = 180°$; thus $D''AD'''$ is a straight line. Similarly, $D'''BD'$ and $D'CD''$ are straight lines. Hence $D'D''D'''$ is a triangle with C, A, B as the mid-points of its sides. Therefore BC is parallel to $D'''D''$, and $B' = B$, $C' = C$. Continuing in this way we see that the four triangles are congruent. Thus the tetrahedron $ABCD$ is such that its opposite edges are equal.

Such tetrahedrons may be constructed by taking any triangle $D'D''D'''$ which has only acute angles and reversing the above process.

3505 [1931, 408] *Proposed by J. Rosenbaum, Milford, Conn.*

Under what condition will the lines joining the vertices of a tetrahedron with the points of contact of the opposite faces and the insphere be concurrent?

Solution by the Proposer.

Let the vertices of the tetrahedron be A_1, A_2, A_3, A_4; the faces opposite them F_1, F_2, F_3, F_4; and the points of contact of these faces with the insphere O_1, O_2, O_3, O_4.

From the assumption that A_1O_1, A_2O_2, A_3O_3 and A_4O_4 are concurrent, follows that three lines such as A_1O_3, A_3O_1 and A_2A_4 are also concurrent. Accordingly, draw A_1O_3 and A_3O_1 each meeting A_2A_4 at P. Because O_1, O_2, etc. are the points of contact, the three segments A_1O_2, A_1O_3 and A_1O_4 are equal. From this it follows that the triangles $O_1A_2A_4$ and $O_3A_2A_4$ are congruent; from which in turn it follows that the triangles O_1A_2P and O_3A_2P are congruent; also the segments O_1P and O_3P are equal.

From the concurrency of the lines of the problem it can further be proved that four masses, m_1, m_2, m_3, m_4 can be placed at the vertices, A_1, A_2, A_3, A_4 such that O_1 is the center of mass of the masses at the vertices of F_1; O_2 of the masses at the vertices of F_2; etc., and also that the point P mentioned above is the center of m_2 and m_4.

Considering the face F_3, since O_3 is the center of m_4, m_1 and m_2, and since P is the center of m_2 and m_4,

$$A_1O_3/O_3P = (m_2 + m_4)/m_1. \tag{1}$$

In the same way, by considering the face F_1

$$A_3O_1/O_1P = (m_2 + m_4)/m_3. \tag{2}$$

Dividing (1) by (2), and taking account of the equality of O_3P and O_1P, we have

$$A_1O_3/A_3O_1 = m_3/m_1. \tag{3}$$

Replacing A_1O_3 and A_3O_1 by A_1O_4 and A_3O_4, equation (3) becomes

$$A_1O_4/A_3O_4 = m_3/m_1. \tag{4}$$

Considering now the face F_4 and drawing A_2O_4 cutting A_1A_3 at Q, we obtain, since Q is the center of m_1 and m_3,

$$A_1Q/QA_3 = m_3/m_1. \tag{5}$$

From (4) and (5),

$$A_1Q/QA_3 = A_1O_4/O_4A_3. \tag{6}$$

From this last equation follows that O_4Q, which is the same line as A_2O_4, bisects the angle $A_1O_4A_3$. Similarly A_1O_4 bisects angle $A_1O_4A_2$. Hence the angles $A_1O_4A_2$, $A_2O_4A_3$, and $A_3O_4A_1$ are 120 degrees.

This result applies also to the corresponding angles in the other faces.

Thus is obtained the conclusion: In order that the lines of the problem may be concurrent it is necessary that the point of contact in each face be the isogonic point of the face. It can be proved that this condition is also sufficient.

The construction of such a tetrahedron from cardboard will show that it does not need to be regular.

Also solved by William Hoover and A. Pelletier.

3514 [1931, 462]. *Proposed by J. P. Ballantine, University of Washington.*

Let D_n denote a determinant of order n whose elements are all zeros and ones, and which has 2 ones and n-2 zeros in every row and column. Show that:

(a) For every n, $D_n = \pm 2^m$, where m and n are both even or both odd, or $D_n = 0$.

(b) If $D_5 = 0$, then two rows are identical, and conversely.

(c) If two rows of D_n are identical, then two columns are alike, and conversely.

(d) $3m \leqq n$.

(e) Show that property (b) does not hold except for D_2, D_3, and D_5.

Solution by Dwight F. Gunder, Colorado Agricultural College

By rearrangement of the order of the rows and the columns D_n reduces to;

$$D_n = \pm \begin{vmatrix} A_1 & 0 & 0 \cdot \cdot & 0 \\ 0 & A_2 & 0 \cdot \cdot & 0 \\ 0 & 0 & A_3 \cdot \cdot & 0 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \cdot \cdot & A_m \end{vmatrix}$$

where the A_k are square arrays of order n_k ($n_k>1$) with elements $a_{1,1}=a_{n_k,n_k}=1$, $a_{i+1,\,i}=a_{i,i+1}=1$ ($i=1, 2, \cdots, n_k-1$) and all other elements are zero.

Any A_k is at once seen to simplify to B_k in which $b_{i,i}=(-1)^{i+1}$ for $i=1, \cdots, n_k-1$, and b_{n_k,n_k} equals 0 or 2 according as n_k is even or odd, $b_{i,i+1}=1$ ($i=1, \cdots, n_k-1$), and all other elements are zero. This B_k expands at once and we have:

$$A_k = B_k = 0 \quad \text{for } n_k \equiv 0 \bmod 2,$$

$$A_k = B_k = 2 \quad \text{for } n_k \equiv 1 \bmod 4,$$

$$A_k = B_k = -2 \quad \text{for } n_k \equiv 3 \bmod 4.$$

Further, if n is odd $D_n \neq 0$ only when m and every n_k are odd, and if n is even, $D_n \neq 0$ only when m is even and every n_k is odd.

Hence,

(a) For n odd $D_n = \pm 2^m$ where m is odd, or $D_n = 0$, and for n even $D_n = \pm 2^m$ where m is even, or $D_n = 0$.

(b) After the first reduction above, D_5 will have only one A_k (of order five) or will have two A_k, one of order three and the other of order two. In the first case, $D_5 \neq 0$, and in the second case $D_5 = 0$ and by definition of an A_k of order two we find that two rows, and two columns, of D_5 are identical. The converse is obvious.

(c) If two rows of D_n are identical the two ones must fall in corresponding columns, hence two columns are identical, and conversely.

(d) Since to be different from zero an A_k must have an order $n_k \geqq 3$, then obviously

$$n = \sum_{k=1}^{k=m} n_k \geqq 3m.$$

(e) By (b), $D_5 = 0$ necessitates two identical rows, and obviously $D_2 = 0$, and $D_3 = 0$ do likewise; but any $D_n = 0$ ($n = 4$ or $n > 5$) may be due to the appearance of an A_k of order $4p$ whose vanishing does not necessitate two identical rows in D_n. The converse part of (b) is of course always true.

Note by the Editors

$$D_3 = \pm \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = \mp 2.$$

Hence D_3 cannot be zero, and the part of (e) referring to D_3 seems to have no meaning; and for D_2 it is trivial.

Solutions were received from the following contributors after selections for publication had been made: William Hoover 291, 313; R. A. Johnson, 3458; A. Hershfield, 232; F. Underwood, 309, 313; H. S. Uhler, 313; L. B. Haughwout, 313; and V. F. Ivanoff, 313.

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NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Wesleyan University has conferred an honorary doctorate of science on Professor Henry S. White of Vassar College.

Professor E. R. Hedrick, of the University of California at Los Angeles, has been appointed chairman of a committee on the teaching of mathematics in the colleges and universities of the North Central Association. Members of this committee include Professor H. E. Slaught of the University of Chicago, and Professor Dunham Jackson, of the University of Minnesota.

At the ninetieth meeting of the American Association for the Advancement of Science, which was held in Syracuse from June 20 to June 25, the following addresses, papers, and report of interest to mathematicians were presented:

A general session of the meeting on the teaching of mathematics was addressed by Professor W. B. Carver of Cornell University, Editor-in-chief of this Monthly, and by Professor E. R. Hedrick. Professor Carver pointed out some of the contributions of mathematics to science, and indicated how certain branches of mathematics which at first interested only mathematicians became later useful tools *for physicists and engineers*. He also emphasized the need for inspired science text-books and teachers of mathematics. Professor Hedrick made a plea for closer cooperation between mathematicians and educationists, and urged that state laws be passed requiring teachers of science to have had at least one college course in the science they teach.

Papers were read before Section A (mathematics) by Professors H. M. Gehman of the University of Buffalo, W. A. Hurwitz of Cornell University, and J. A. Shohat of the University of Pennsylvania. Professor Gehman developed several theorems on homeomorphic geometry, Professor Hurwitz discussed logical foundations for groups and fields, and Professor Shohat considered a number of problems in interpolation.

Section A met with section K (social and economic sciences) for a session on

mathematical statistics. In a paper by Professor C. F. Roos it was shown how the effect of past prices and the memory of past purchases lead to a demand function containing an integral either of past prices or past purchases, the two forms being equivalent. A report on mathematics needed in the social sciences, adopted by the Social Science Research Council after the deliberations of a committee, was read by Doctor Mordecai Ezekiel. Sections A and K (with the affiliated Econometric Society) of the Association approved the recommendations in this report that in preparation for work in the social sciences undergraduates should study logarithms, graphic methods, interpolation, equations and forms of simple curves, probability, and the elements of differential and integral calculus and curve fitting.

The following names were omitted from the list of doctorates conferred during 1931 as published in the June-July issue of the MONTHLY:

O. W. Albert, Washington (Seattle), June, minor in astronomy, *Relations between the projective and metric differential geometries of surfaces.*

T. C. Bumer, Ohio State, August, *Dynamical systems with hysteresis effects of the Fredholm iype.*

Richard S. Burington, Ohio State University, June, *An invariantive classification of plane cubic curves under the affine group.*

L. S. Kennison, California Institute, June, minor in mathematical physics, *Linear functional equations on a composite range.*

Professor Oliver Dimon Kellog of Harvard University, a charter member of the Association, died suddenly while mountain climbing near Greenville, Maine, August 27, 1932, at the age of 54 years. He was connected with the department of mathematics at the University of Missouri, 1905–1920, and at Harvard University, 1920 to the time of his death. A complete statement of his scientific career will appear in due time.

Professor Ernest Julius Wilczynski died in Denver, Colorado, on September 15, 1932 at the age of 56. He had been incapacitated for active service since the summer of 1923. He was a charter member of the Mathematical Association of America and a loyal supporter of its aims and activities, as vice-president, council member, and contributor to the MONTHLY. He had been connected with the department of mathematics in the University of California, 1898–1907, University of Illinois, 1907–1910, and the University of Chicago, 1910 to the time of his death, with the title of Professor Emeritus of Mathematics in recent years.

THE SIXTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The sixteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of California at Los Angeles on Monday and Tuesday, August 29 and 30, 1932, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred and ten were present at the meetings, including the following ninety-two members of the Association:

BEATRICE AITCHISON, Johns Hopkins University
O. W. ALBERT, University of Redlands
L. D. AMES, University of Southern California
CLARA L. BACON, Goucher College
J. P. BALLANTINE, University of Washington
M. A. BASOCO, University of Nebraska
HARRY BATEMAN, California Institute of Technology
MAY M. BEENKEN, State Teachers College, Oshkosh, Wisconsin
CLIFFORD BELL, University of California at Los Angeles
E. T. BELL, California Institute of Technology
B. A. BERNSTEIN, University of California
R. W. BOLTON, Glendale, Calif.
J. L. BOTSFORD, California Institute of Technology
W. A. BRATTON, Whitman College
A. L. BUCKMAN, University of California
J. H. BUSHEY, Hunter College
JESSIE R. CAMPBELL, Hollywood Junior College
W. B. CARVER, Cornell University
A. H. CLIFFORD, California Institute of Technology
MYRTIE COLLIER, Los Angeles
R. H. CONLEE. Fairfield, Iowa
LENNIE P. COPELAND, Wellesley College
N. A. COURT, University of Oklahoma
P. H. DAUS, University of California at Los Angeles
G. G. ENTZ, Hollywood, Calif.
H. J. ETTLINGER, University of Texas
RAYMOND GARVER, University of California at Los Angeles
HARRIET E. GLAZIER, University of California at Los Angeles
F. L. GRIFFIN, Reed College
LOIS W. GRIFFITHS, Northwestern University
W. L. HART, University of Minnesota
E. R. HEDRICK, University of California at Los Angeles
J. D. HILL, Brown University
JEWELL C. HUGHES, Hunter College
G. H. HUNT, University of California at Los Angeles
J. W. HURST, Montana State College
C. A. HUTCHINSON, University of Colorado
M. H. INGRAHAM, University of Wisconsin
DUNHAM JACKSON, University of Minnesota
C. G. JAEGER, Pomona College
GLENN JAMES, University of California at Los Angeles
C. M. JENSEN, Macalester College
MARY N. KEITH, University of Redlands
D. H. LEHMER, Altadena, Calif.
D. N. LEHMER, University of California
H. B. LEONARD, University of Arizona
JACK LEVINE, Princeton University
G. R. LIVINGSTON, State Teachers College, San Diego, Calif.
DECA LODWICK, High School, Long Beach, Calif.
MAYME I. LOGSDON, University of Chicago
ADA A. MCCLELLAN, High School, Long Beach, Calif.
W. H. MCEWEN, Mount Allison University
JAMES MCGIFFERT, Rensselaer Polytechnic Institute
RUTH G. MASON, Berkeley, Calif.
W. E. MASON, University of California at Los Angeles
KATE M. MEEK, High School, Pasadena, Calif.
A. D. MICHAL, California Institute of Technology
E. L. MICKELSON, State Teachers College, Silver City, New Mexico
W. E. MILNE, University of Oregon
C. V. NEWSOM, University of New Mexico
W. B. ORANGE, Los Angeles Junior College
F. W. OWENS, Pennsylvania State College